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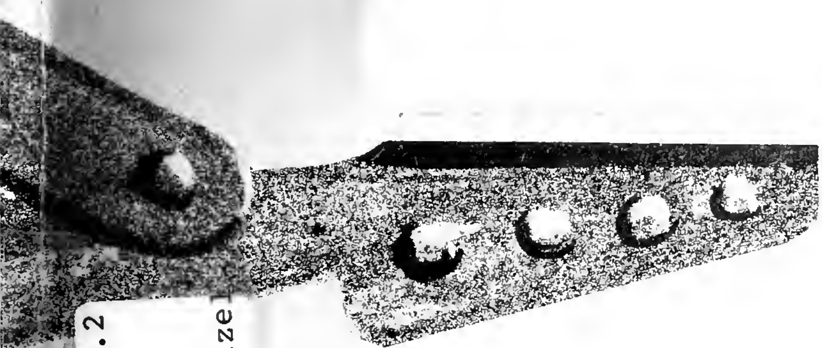

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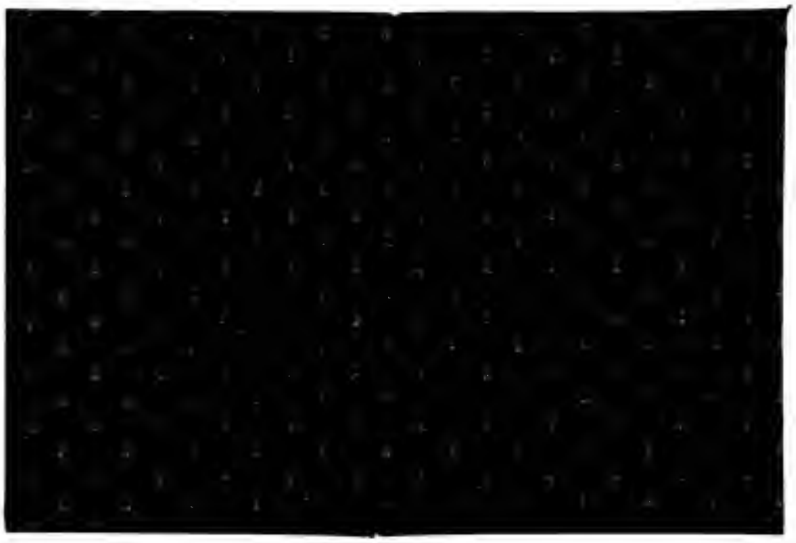

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Sharp Upper and Lower Bounds on the Length of General Davenport-Schinzel Sequences

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ABSTRACT

We obtain sharp upper and lower bounds on the maximal length $\lambda_s(n)$ of (n, s)-Davenport-Schinzel sequences, i.e. sequences composed of n symbols, having no two adjacent equal elements, and containing no alternating subsequence of length $s+2$. We show that (i) $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$, (ii) for $s > 4$, $\lambda_s(n) \le n \cdot 2^{(\alpha(n))^{\frac{s-2}{2}} + C_s(n)}$ if s is even and $\lambda_s(n) \le n \cdot 2^{(\alpha(n))^{\frac{s-3}{2}} \log(\alpha(n)) + C_s(n)}$ if s is odd, where $C_s(n)$ is a function of $\alpha(n)$ and s, asymptotically smaller than the main term, and finally (iii) for even values of $s > 4$, $\lambda_s(n) = \Omega\left(n \cdot 2^{K_s(\alpha(n))^{\frac{s-2}{2}} + Q_s(n)}\right)$ where $K_s = \frac{1}{\frac{(s-2)}{2}!}$ and Q_s is a polynomial in $\alpha(n)$ of degree at most $\frac{s-4}{2}$.

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1 Introduction:

In this paper we obtain optimal bounds for the maximal length $\lambda_4(n)$ of an $(n, 4)$ *Davenport-Schinzel Sequence* (a $DS(n, 4)$ sequence in short), and then extend them to improve and almost tighten the lower and upper bounds for $\lambda_s(n)$, $s > 4$. A $DS(n, s)$ sequence, $U = (u_1, \ldots, u_m)$ is a sequence composed of n distinct symbols which satisfies the following two conditions:

1. $\forall i < m$, $u_i \neq u_{i+1}$.
2. There do not exist $s+2$ indices $1 \leq i_1 < i_2 \cdots < i_{s+2} \leq m$ such that

$$u_{i_1} = u_{i_3} = u_{i_5} = \cdots = a,$$

$$u_{i_2} = u_{i_4} = u_{i_6} = \cdots = b$$

and $a \neq b$.

We refer to s as the *order* of the sequence U. We write $|U| = m$ for the *length* of the sequence U; thus

$$\lambda_s(n) = \max\{|U| : U \text{ is a } DS(n, s) \text{ sequence}\}.$$

Davenport-Schinzel sequences have turned out to be of central significance in computational and combinatorial geometry and related areas, and have many applications in diverse areas including motion planning, shortest path, visibility, transversals, Voronoi diagrams, arrangements and many more; see [At], [BS], [Cl], [CS], [ES], [GS], [GSS], [HS], [KS], [LS], [OSY], [PS], [PSS], [SS], [WS]. It is shown [At] that $DS(n, s)$ sequences provide a combinatorial characterization of the lower envelope of n continuous univariate functions, each pair of which intersect in at most s points. Thus $\lambda_s(n)$ is the maximum number of connected portions of the graph of n such functions which constitute their lower envelope. Since minimization of functions is a central operation in many geometric and other combinatorial problems, sharp estimates of $\lambda_s(n)$ yield sharp and often near-optimal bounds for the complexity of these problems. This, combined with the highly non-trivial and surprising form of the bounds on $\lambda_s(n)$, as given below, makes Davenport-Schinzel sequences a very powerful and versatile tool.

The problem of estimating $\lambda_s(n)$ has been studied by several authors [DS], [Da], [RS], [Sz], [At], [HS], [Sh1], [Sh2]. It is easy to show that $\lambda_1(n) = n$ and

$\lambda_2(n) = 2n - 1$. Hart and Sharir [HS] have shown that $\lambda_3(n) = \Theta(n\alpha(n))$. Here $\alpha(n)$ is a functional inverse of Ackermann's function and is very slowly growing. For higher order sequences, the best known upper bounds are due to Sharir [Sh1] and have the following form:

$$\lambda_s(n) = O(n\alpha(n)^{O(\alpha(n)^{s-3})}) \qquad \text{for } s \geq 4$$

and the best known lower bounds are [Sh2]:

$$\lambda_{2s+1}(n) = \Omega(n(\alpha(n))^s) \qquad \text{for } s \geq 2$$

Thus for $s \geq 4$ there has still been a gap between the lower and upper bounds for $\lambda_s(n)$. In this paper we first establish tight upper and lower bounds for $\lambda_4(n)$ and then obtain sharp, and almost tight, upper and lower bounds for $\lambda_s(n)$ for higher values of s, by generalizing the techniques used in the case of $\lambda_4(n)$.

The main results of this paper are as follows:

(i) The maximal length of a $DS(n, 4)$ sequence is

$$\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)}).$$

(ii) An upper bound on the maximal length of a $DS(n, s)$ sequence is

$$\lambda_s(n) \leq \begin{cases} n \cdot 2^{(\alpha(n))^{\frac{s-2}{2}} + C_s(n)} & \text{if } s \text{ is even} \\ n \cdot 2^{(\alpha(n))^{\frac{s-3}{2}} \log(\alpha(n)) + C_s(n)} & \text{if } s \text{ is odd} \end{cases}$$

where $C_s(n)$ is a function of $\alpha(n)$ and s. For a fixed value of s, $C_s(n)$ is asymptotically smaller than the first term of the exponent and therefore for sufficiently (and extremely) large values of n the first term of the exponent dominates.

(iii) A lower bound on the maximal length of a $DS(n, s)$ sequence of an even order is

$$\lambda_s(n) = \Omega\left(n \cdot 2^{K_s(\alpha(n))^{\frac{s-2}{2}} + Q_s(n)}\right)$$

where $K_s = \frac{1}{\frac{(s-2)}{2}!}$ and $Q_s(n)$ is a polynomial in $\alpha(n)$ of degree at most $\frac{s-4}{2}$.

Thus our lower and upper bounds are much closer than the previous bounds although they are still not tight. For even s they are almost identical except for the constant K_s and the lower order additive terms $C_s(n)$, $Q_s(n)$, appearing in the exponents. For odd s the gap is more "substantial".

The proofs are fairly complicated and involve a lot of technical details. For the sake of exposition, we first present the derivation of the tight bounds for $\lambda_4(n)$, which gives the general flavor of the techniques used in establishing the bounds, but is relatively much simpler. Then we generalize these techniques for higher values of s. Another reason for considering $\lambda_4(n)$ separately is that we solve the recurrence relation that gives an upper bound for $\lambda_4(n)$ in a slightly more "efficient" way, which enables us to get tight bounds, while for general values of s, where no such refinement could be obtained, the proofs are slightly different. The paper is organized as follows: In section 2, we give the upper bounds for $\lambda_4(n)$; in section 3, we construct a class of $DS(n, 4)$ sequences and prove that their length is $\Omega(n \cdot 2^{\alpha(n)})$; in section 4, we prove the upper bounds for general values of s and finally in section 5, we establish our lower bounds for higher values of s. The proofs introduce and exploit several variants of Ackermann's functions. A large technical part of the proofs involves derivation of various proprties of these functions. These derivations have been grouped into several appendices at the end of the paper.

2 The Upper Bound for λ_4(n):

The best previously known upper bound for $\lambda_4(n)$ was $O(n \cdot \alpha(n)^{O(\alpha(n))})$, as follows from [Sh1]. In this section we improve his bound by showing that $\lambda_4(n) = O(n \cdot 2^{\alpha(n)})$.

2.1 Decomposition of DS-sequences into Chains:

We begin by reviewing some definitions and facts from [Sh1].

Definition: Let U be a $DS(n,s)$ sequence, and let $1 \leq t < s$. A *t-chain* c is a contiguous subsequence of U which is a Davenport-Schinzel sequence of order t.

Given n, s, t and U as above, we partition U into disjoint t-chains, proceeding from left to right in the following inductive manner. Suppose that the initial portion $(u_1, \ldots, u_j)$ of U has already been decomposed into t-chains. The next t-chain in our partitioning is then the largest subsequence of U of the form $(u_{j+1}, \ldots, u_k)$ which is still a Davenport-Schinzel sequence of order t. We refer to this partitioning as the *canonical decomposition* of U into t-chains, and let $m = m_t(U)$ denote the number of t-chains in this decomposition.

The problem of obtaining good upper bounds for the quantities

$$\mu_{s,t}(n) = \max \{ m_t(U) : U \text{ is a } DS(n,s) \text{ sequence} \}$$

seems quite hard for general s and t. Sharir [Sh1] has proved the following result:

Lemma 2.1 *$\mu_{s,s-1}(n) \leq n$ and $\mu_{s,s-2}(n) \leq 2n - 1$.*

The above result shows, in particular, that a $DS(n,4)$ sequence can be decomposed into at most $2n-1$ 2-chains.

Lemma 2.2 *Given a $DS(n,4)$ sequence U composed of m 2-chains, we can construct another $DS(n,4)$ sequence U' composed of m 1-chains such that $|U'| \geq \frac{1}{2}(|U| - m)$.*

Proof: Replace each 2-chain c by a 1-chain c' composed of the same symbols of c in the order of their leftmost appearances in c. Since $\lambda_2(n) = 2n - 1$, we have

$|c'| \geq \frac{1}{2}|c| + \frac{1}{2}$. Take the concatenation of all these 1-chains, erasing each first element of a chain that is equal to its preceding element. The resulting sequence U' is clearly a $DS(n,4)$ sequence composed of at most m 1-chains, whose length is $|U'| \geq \frac{1}{2}|U| + \frac{m}{2} - m = \frac{1}{2}(|U| - m)$.

□

Definition: Let n, m and s be positive integers. We denote by $\Psi_s^t(m,n)$ the maximum length of a $DS(n,s)$ sequence composed of at most m t-chains. If $t = 1$, we denote it by $\Psi_s(m, n)$ also.

Corollary 2.3 $\lambda_4(n) \leq 2\Psi_4(2n-1, n) + 2n - 1$.

Proof: The proof directly follows from Lemma 2.1 and 2.2.

□

The main result of this section is that $\Psi_4(m, n) = O((m+n) \cdot 2^{\alpha(m)})$. This upper bound for $\Psi_4(m, n)$ in conjunction with Corollary 2.3 gives the desired upper bound for $\lambda_4(n)$.

2.2 Some Properties of Ackermann's Function and Related Functions:

Before proving the main result, we prove certain properties of Ackermann's function and some auxiliary functions which we need in establishing the desired upper bound. For a more basic review of Ackermann's function see [HS].

We first review the definition of Ackermann's function. Let $\mathcal{N}$ be the set of positive integers $1, 2, \ldots$. Given a function g from a set into itself, denote by $g^{(s)}$ the composition $g \circ g \circ \cdots \circ g$ of g with itself s times, for $s \in \mathcal{N}$. Define inductively a sequence $\{A_k\}_{k=1}^{\infty}$ of functions from $\mathcal{N}$ into itself as follows:

$$\begin{aligned} A_1(n) &= 2n, \\ A_k(n) &= A_{k-1}^{(n)}(1), \qquad k \geq 2 \end{aligned}$$

for all $n \in \mathcal{N}$. Note that for all $k \geq 2$, the function A_k satisfies

$$\begin{aligned} A_k(1) &= 2, \\ A_k(n) &= A_{k-1}(A_k(n-1)), \qquad n \geq 2. \end{aligned}$$

In particular $A_2(n) = 2^n$ and $A_3(n) = 2^{2^{\cdot^{\cdot^{2}}}}$ with n 2's in the exponential tower. Finally, Ackermann's function is defined as:

$$A(n) = A_n(n).$$

For any (weakly) monotone function $g : \mathcal{N} \to \mathcal{N}$ its functional inverse $\gamma(n)$ is defined as

$$\gamma(n) = \min\{\, j \,:\, g(j) \geq n\}$$

Let α_k and α denote the functional inverses of A_k and A, respectively. Then, for all $n \in \mathcal{N}$, the functions $\alpha_k(n)$ are given by the following recursive formula:

$$\alpha_k(n) = \min\{\, s \geq 1 \,:\, \alpha_{k-1}^{(s)}(n) = 1\};$$

that is, $\alpha_k(n)$ is the number of iterations of α_{k-1} needed to go from n to 1.

All the functions $\alpha_k(n)$ are non-decreasing, and converge to infinity with their argument. The same holds for $\alpha(n)$ too, which grows more slowly than any of the $\alpha_k(n)$.

The following property, which follows immediately from the above definitions, will be used in the sequel

$$\alpha_{\alpha(n)}(n) \leq \alpha(n). \tag{2.1}$$

In the following lemmas we prove some more properties of $A_k(n)$ and other auxiliary functions $\beta_k(n)$ defined below. The proof of these lemmas are given in the Appendix 1 as these proofs are somewhat technical and they are not required in the proofs of the main lemmas.

Lemma 2.4 *For all* $k \geq 1$, $A_k(2) = 4$ *and* $A_k(3) \geq 2k$.

The above lemma implies that $\alpha_k(4) = 2$ and $\alpha_k(k) \leq \alpha_k(2k) \leq 3$ for all $k \geq 1$. We use these results in the next lemma.

Lemma 2.5 *For all* $n \geq 1$, $\alpha_{\alpha(n)+1}(n) \leq 4$.

Lemma 2.6 *For all $k \geq 4$ and $s \geq 3$,*

$$2^{A_k(s)} \leq A_{k-1}(\log(A_k(s)))$$

Lemma 2.7 *Let $\xi_k(n)$ be $2^{\alpha_k(n)}$. Then for $k \geq 3$, $n \geq A_{k+1}(4)$,*

$$\min\{\, s' \geq 1 \,:\, \xi_k^{(s')}(n) \,\leq\, A_{k+1}(4)\} \,\leq 2 \cdot \alpha_{k+1}(n) - 2$$

We define a sequence of functions $\beta_k(n)$ which are related to the inverse Ackerman functions as follows:

$$\begin{aligned} \beta_1(n) &= \alpha_1(n) \\ \beta_2(n) &= \alpha_2(n) \\ \beta_k(n) &= \min\{\, s \geq 1 \,:\, (\alpha_{k-1} \cdot \beta_{k-1})^{(s)}(n) \leq 64\} \end{aligned}$$

The functions $\beta_k(n)$ are non-decreasing, and converge to infinity with their argument. Note that

$$\beta_3(n) = \min\{\, s \geq 1 \,:\, \left(\lceil \log \rceil^2\right)^{(s)}(n) \,\leq\, 64\}$$

In the next lemma we give an upper bound on $\beta_k(n)$ which shows that they grow at the same rate as $\alpha_k(n)$.

Lemma 2.8 *For all $k \geq 1$, $n \geq 2$, $\beta_k(n) \leq 2\alpha_k(n)$.*

2.3 Upper Bound For $\Psi_4(\mathbf{m}, \mathbf{n})$:

In this subsection we establish an upper bound on the maximal length $\Psi_4(m, n)$ of an $(n, 4)$ Davenport Schinzel sequence composed of at most m 1-chains. The following lemma is a refinement of proposition 4.1 of [Sh1].

Lemma 2.9 *Let $m, n \geq 1$, and $1 < b < m$ be integers. Then for any partitioning $m = \sum_{i=1}^{b} m_i$ with $m_1, \ldots, m_b \geq 1$ there exist integers $n^*, n_1, n_2, \ldots, n_b \geq 0$ such that*

$$n^* + \sum_{i=1}^{b} n_i = n,$$

and

$$\Psi_4(m,\, n) \le \sum_{i=1}^{b} \Psi_4(m_i, n_i) \,+\, 2\Psi_4(b,\, n^\star) \,+\, \Psi_3(m,\, 2n^\star) + 3m \qquad (2.2)$$

Proof: Let U be a $DS(n,4)$-sequence consisting of at most m 1-chains $c_1, \ldots, c_m$ such that $|U| = \Psi_4(m,\, n)$, and let $m = \sum_{i=1}^{b} m_i$ be a partitioning of m as above. Partition the sequence U into b *layers* (i.e. contiguous subsequences) $L_1, \ldots, L_b$ so that the layer L_i consists of m_i 1-chains. Call a symbol a *internal* to layer L_i if all the occurrences of a in U are within L_i. A symbol will be called *external* if it is not internal to any layer. Suppose that there are n_i internal symbols in layer L_i and $n^\star$ external symbols (thus $n^\star + \sum_{i=1}^{b} n_i = n$).

To estimate the total number of occurrences in U of symbols that are internal to L_i, we proceed as follows. Erase from L_i all external symbols. Next scan L_i from left to right and erase each element which has become equal to the element immediately preceding it. This leaves us with a sequence $L_i^\star$ which is clearly a $DS(n_i,\, 4)$ sequence consisting of at most m_i 1-chains, and thus its length is at most $\Psi_4(m_i,\, n_i)$. Moreover, if two equal internal elements in L_i have become adjacent after erasing the external symbols, then these two elements must have belonged to two distinct 1-chains, thus the total number of deletions of internal symbols is at most $m_i - 1$.

Hence, summing over all layers, we conclude that the total contribution of internal symbols to $|U|$ is at most

$$m - b + \sum_{i=1}^{b} \Psi_4(m_i,\, n_i).$$

We estimate the total number of occurrences of external symbols in two parts. For each layer L_i, call an external symbol a *middle symbol* if it neither starts in L_i nor ends in L_i. If an external symbol is not a middle symbol, call it an *end symbol*. An external symbol appears as an end symbol exactly in two layers. First we estimate the contribution of middle symbols. For each layer L_i erase all internal symbols and end symbols and if necessary, also erase each occurrence of a middle symbol which has become equal to the element immediately preceding it. The above process deletes at most $m_i - 1$ middle symbols. Let us denote the resultant sequence by $L_i^\star$.

We claim that the $L_i^\star$ is a $DS(p_i, 2)$ sequence, where p_i is the number of distinct symbols in $L_i^\star$. Suppose the contrary; then $L_i^\star$ has a subsequence of the form

$$a \dots b \dots a \dots b$$

where a and b are two distinct symbols of $L_i^\star$. But they are middle symbols, i.e. each appears in a layer before $L_i^\star$ as well as in a layer after $L_i^\star$. This implies that U has a subsequence of the form

$$(b \dots a) \dots a \dots b \dots a \dots b \dots (b \dots a)$$

in which each of the first and last pairs may appear in reverse order. But this alternation of length ≥ 6 contradicts the fact that U is a $DS(n, 4)$ sequence. Therefore, $L_i^\star$ is a $DS(p_i, 2)$ sequence. Thus, the concatenation of all sequences $L_i^\star$, with the additional possible deletions of any first element of $L_i^\star$ which happens to be equal to the last element of $L_{i-1}^\star$, is a $DS(n^\star, 4)$ sequence V composed of b 2-chains, and it follows from (2.2) that we can replace this sequence by another $DS(n^\star, 4)$ sequence $V^\star$ composed of b 1-chains so that $|V| \leq 2|V^\star| + b$. Hence, the contribution of middle symbols to $|U|$ is at most

$$2\Psi_4(b, n^\star) + m + b$$

Now, we consider the contribution of end symbols. For each layer L_i, erase all internal symbols and middle symbols and if necessary also erase each occurrence of an end symbol if it is equal to the element immediately preceding it. We erase at most $m_i - 1$ end symbols. Let us denote the resultant sequence by $L_i^\#$. Let q_i be the number of distinct symbols in $L_i^\#$. We claim that $L_i^\#$ is a $DS(q_i, 3)$ sequence. Indeed, if this were not the case, $L_i^\#$ would have contained an alternating subsequence of the form

$$a \dots b \dots a \dots b \dots a$$

Since b is an external symbol, it also appears in a sequence $L_j^\#$ other than $L_i^\#$. But then U has an alternation of length six which is impossible. Hence, $L_i^\#$ is a $DS(q_i, 3)$ sequence consisting of m_i 1-chains, so its length is at most $\Psi_3(m_i, q_i)$. Summing over all the layers, the contribution of the end symbols is at most

$$m + \sum_{i=1}^{b} \Psi_3(m_i, q_i) \leq m + \Psi_3(m, \sum_{i=1}^{b} q_i)$$

But an external symbol appears as an end symbol only in two layers, therefore $\sum_{i=1}^{b} q_i = 2n^\star$. Hence, the total contribution of external symbols is at most

$$2m + b + 2\Psi_4(b,\, n^\star) + \Psi_3(m,\, 2n^\star)$$

Thus, we obtain the asserted inequality:

$$\Psi_4(m,\, n) \le \sum_{i=1}^{b} \Psi_4(m_i,\, n_i) + 3m + \Psi_3(m, 2n^\star) + 2\Psi_4(b,\, n^\star)$$

□

Lemma 2.10 *Let $n,\, m \ge 1$, $k \ge 2$. Then*

$$\Psi_4(m,\, n) \le (\frac{15}{2} \cdot 2^k - 4k - 11)m \cdot \alpha_k(m) \cdot \beta_k(m) + (\frac{21}{2} \cdot 2^k - 4k - 8) \cdot n \quad (2.3)$$

Proof: We use equation 2.2 repeatedly to obtain the above upper bounds for $k = 2, 3, \ldots$. At each step we choose b appropriately and estimate $\Psi_4(b,\, n^\star)$ using a technique similar to that in [HS] and [Sh1]. At the k^{th} step we refine the bound of $\Psi_3(m,\, 2n^\star)$ using the inequality

$$\Psi_3(m,\, n) \le 4km \cdot \alpha_k(m) + 2kn$$

obtained in [HS].

We proceed by double induction on k and m. Initially $k = 2$, $m \ge 1$, and $\beta_k(m) = \alpha_k(m) = \lceil \log m \rceil$. Choose $b = 2$, $m_1 = \left\lfloor \frac{m}{2} \right\rfloor$, $m_2 = \left\lceil \frac{m}{2} \right\rceil$ in the equation (2.2):

$$\Psi_4(b,\, n^\star) = \Psi_4(2,\, n^\star) = 2n^\star$$

for all $n^\star$, and $\Psi_3(m, 2n^\star)$ is

$$\le 8m \lceil \log m \rceil + 8n^\star$$

so the equation (2.2) becomes:

$$\begin{aligned}\Psi_4(m,\, n) &= \Psi_4\left(\left\lfloor \frac{m}{2} \right\rfloor, n_1\right) + \Psi_4\left(\left\lceil \frac{m}{2} \right\rceil, n_2\right) + 12n^\star + 3m + 8m \lceil \log m \rceil \\ &\le \Psi_4\left(\left\lfloor \frac{m}{2} \right\rfloor, n_1\right) + \Psi_4\left(\left\lceil \frac{m}{2} \right\rceil, n_2\right) + 12n^\star + 11m \lceil \log m \rceil\end{aligned}$$

where $n = n_1 + n_2 + n^\star$. The solution of the above equation is easily seen to be

$$\begin{aligned}
\Psi_4(m, n) &\le 11m(\lceil \log m\rceil)^2 + 12n \\
&= 11m \cdot \alpha_2(m) \cdot \beta_2(m) + 12n \\
&\le (\frac{15}{2} \cdot 2^k - 4k - 11) \cdot m\alpha_k(m) \cdot \beta_k(m) + (\frac{21}{2} \cdot 2^k - 4k - 8) \cdot n
\end{aligned}$$

For $k > 2$ and $m \le 64$ the inequality holds because $\Psi_4(m, n) \le 64n$ which is less than the right hand side of the inequality.

For $k > 2$ and $m > 64$, assume that the inductive hypothesis is true for all $k' < k$ and $m' \ge 1$ and for $k' = k$ and $m' < m$; choose

$$t = \left\lceil \frac{\alpha_{k-1}(m) \cdot \beta_{k-1}(m)}{\alpha_k(m)} \right\rceil, \text{ and } b = \left\lfloor \frac{m}{t} \right\rfloor.$$

For $m > 64$ and $k > 2$, $\alpha_k(m) > 2$ and $\alpha_{k-1}(m) \cdot \beta_{k-1}(m) \le \lceil \log m\rceil^2$; thus

$$t \le \left\lceil \frac{1}{2}\lceil \log m\rceil^2 \right\rceil < m - 1.$$

Suppose $m = b \cdot t + r$, then for the first r layers $L_1, \ldots, L_r$ choose $m_i = t + 1$ and for the remaining layers choose $m_i = t$; therefore $m_i \le t + 1 < m$ for all i.

By induction hypothesis (for $k - 1$ and b) we have

$$\begin{aligned}
\Psi_4(b, n^\star) &\le (\frac{15}{2} \cdot 2^{(k-1)} - 4(k-1) - 11)b \cdot \alpha_{k-1}(b) \cdot \beta_{k-1}(b) \\
&+ (\frac{21}{2} \cdot 2^{(k-1)} - 4(k-1) - 8)n^\star
\end{aligned}$$

$$\begin{aligned}
\text{But} \qquad b &\le \frac{m}{t} = \frac{m}{\left\lceil \frac{\alpha_{k-1}(m)\cdot\beta_{k-1}(m)}{\alpha_k(m)} \right\rceil} \\
&\le \frac{m \cdot \alpha_k(m)}{\alpha_{k-1}(m) \cdot \beta_{k-1}(m)}
\end{aligned}$$

Since clearly $b \le m$, we have

$$\Psi_4(b, n^\star) \le (\frac{15}{2} \cdot 2^{(k-1)} - 4(k-1) - 11) \cdot m\alpha_k(m) + (\frac{21}{2} \cdot 2^{(k-1)} - 4(k-1) - 8) \cdot n^\star$$

Since each $m_i < m$, by inductive hypothesis (for $k - 1$ and m_i) equation (2.3) becomes

$$\Psi_4(m, n) \le 2 \cdot (\frac{15}{2} \cdot 2^{(k-1)} - 4(k-1) - 11) \cdot m\alpha_k(m) +$$

$$2 \cdot (\frac{21}{2} \cdot 2^{(k-1)} - 4(k-1) - 8) \cdot n^{\star} +$$
$$4km \cdot \alpha_k(m) + 4kn^{\star} + 3m +$$
$$\sum_{i=1}^{b} \left((\frac{15}{2} \cdot 2^k - 4k - 11) \cdot m_i \cdot \alpha_k(m_i) \cdot \beta_k(m_i) + (\frac{21}{2} \cdot 2^k - 4k - 8) n_i \right)$$

The value of $\beta_k(m_i)$ can be estimated as follows:

$$\begin{aligned} \beta_k(m_i) &\leq \beta_k(t+1) \\ & \beta_k \left(\left\lceil \frac{\alpha_{k-1}(m) \cdot \beta_{k-1}(m)}{\alpha_k(m)} \right\rceil + 1 \right) \\ &\leq \beta_k \left(\frac{\alpha_{k-1}(m) \cdot \beta_{k-1}(m)}{\alpha_k(m)} + 2 \right) \end{aligned}$$

But for all $m > 4$, $\alpha_k(m) \geq 3$ (and $\alpha_{k-1}(m) \cdot \beta_{k-1}(m) \geq 3$ too) and for $x \geq 3, y \geq 3$, $\frac{x}{y} + 2 \leq x$, therefore

$$\begin{aligned} \beta_k(m_i) &\leq \beta_k(\alpha_{k-1}(m) \cdot \beta_{k-1}(m)) \\ &= \beta_k(m) - 1 \end{aligned}$$

which implies

$$\begin{aligned} \Psi_4(m, n) &\leq (\frac{15}{2} \cdot 2^k - 4k - 11) \cdot \alpha_k(m) \cdot (\beta_k(m) - 1) \cdot \sum_{i=1}^{b} m_i + \\ & (\frac{15}{2} \cdot 2^k - 4k - 11) \cdot m \cdot \alpha_k(m) + \\ & (\frac{21}{2} \cdot 2^k - 4k - 8) \cdot (\sum_{i=1}^{b} n_i + n^{\star}) \\ &= (\frac{15}{2} \cdot 2^k - 4k - 11) m \cdot \alpha_k(m) \beta_k(m) + (\frac{21}{2} \cdot 2^k - 4k - 8) n \end{aligned}$$

because $\sum_{i=1}^{b} m_i = m$ and $n^{\star} + \sum_{i=1}^{b} n_i = n$.

□

Theorem 2.11

$$\Psi_4(m, n) = O((m+n) \cdot 2^{\alpha(m)})$$

Proof: By Lemma 2.8, $\beta_k(m) \leq 2\alpha_k(m)$, therefore

$$\Psi_4(m, n) \leq 2 \cdot (\frac{15}{2} \cdot 2^k - 4k - 11)m \cdot (\alpha_k(m))^2 + (\frac{21}{2} \cdot 2^k - 4k - 8) \cdot n$$

Choose $k = \alpha(m) + 1$. By Lemma 2.5, $\alpha_{\alpha(m)+1}(m) \leq 4$. Substituting this value of k in the above inequality we get

$$\begin{aligned} \Psi_4(m, n) &\leq 2 \cdot \frac{15}{2} \cdot 2^{\alpha(m)+1} \cdot m \cdot 16 + \frac{21}{2} \cdot 2^{\alpha(m)+1} \cdot n \\ &= (480m + 21n) \cdot 2^{\alpha(m)} \end{aligned} \tag{2.4}$$

Therefore,

$$\Psi_4(m, n) = O((m + n) \cdot 2^{\alpha(m)})$$

□

Corollary 2.3 therefore yields:

Theorem 2.12

$$\lambda_4(n) = O(n \cdot 2^{\alpha(n)})$$

3 The Lower Bound for λ_4(n):

In this section we establish the matching lower bound for $\lambda_4(n) = \Omega(n \cdot 2^{\alpha(n)})$ which improves the previous bounds given by [Sh2].

Our construction is based on a doubly inductive process which some what resembles that of [WS]. In this construction we use a sequence of functions $F_k(m)$ which grow faster than $A_k(m)$ but nevertheless asymptotically at the same rate.

3.1 The Functions $\mathbf{F_k(m)}$ and Their Properties:

Define inductively a sequence $\{F_k\}_{k=1}^{\infty}$ of functions from the set $\mathcal{N}$ to itself as follows:

$$
\begin{array}{rcll}
F_1(m) & = & 1, & m \geq 1, \\
F_k(1) & = & (2^k - 1)F_{k-1}(2^{k-1}), & k \geq 2, \\
F_k(m) & = & 2F_k(m-1) \cdot F_{k-1}(F_k(m-1)), & k \geq 2,\ m > 1.
\end{array}
$$

Here are some properties of $F_k(m)$.

(P.1) $F_2(m) = 3 \cdot 2^{m-1} \geq A_2(m)$.

(P.2) Each function $F_k(m)$ is strictly increasing in m for all $k \geq 2$.
Thus $F_k(m) \geq m+1$ and $\rho \cdot F_k(m) \leq F_k(\rho \cdot m)$ $\forall k \geq 2$.

(P.3) $\{F_k(m)\}_{k\geq 1}^{\infty}$ is strictly increasing for a fixed $m \geq 1$.

(P.4) $F_k(m) \geq A_k(m)$ for $k \geq 2$, $m \geq 1$.

(P.5) $2^{F_k(m)} \leq A_k(m+4)$ for $k \geq 3$, $m \geq 1$.

(P.6) Hence, $A_k(m) \leq F_k(m) \leq A_k(m+4)$ $\forall k \geq 3$.

(P.3)-(P.5) are proved in Appendix 2. We will also use an auxiliary sequence $\{N_k\}_{k\geq 1}^{\infty}$ of functions defined on the integers as follows

$$
\begin{array}{rcll}
N_1(m) & = & m, & m \geq 1, \\
N_k(1) & = & N_{k-1}(2^{k-1}), & k \geq 2, \\
N_k(m) & = & 2N_k(m-1) \cdot F_{k-1}(F_k(m-1)) + N_{k-1}(F_k(m-1)) & k \geq 2, m > 1.
\end{array}
$$

3.2 The Sequence $S_k(m)$:

We use a doubly inductive construction similar to that of $\lambda_3(n)$. That is, for each pair of integers $k, m \geq 1$ we define a sequence $S_k(m)$ so that

(i) $S_k(m)$ is composed of $N_k(m)$ symbols;

(ii) $S_k(m)$ is the concatenation of $F_k(m)$ *fans*, where each fan is composed of m distinct symbols $a_1, \ldots, a_m$ and has the form

$$(a_1\, a_2\, \cdots\, a_{m-1}\, a_m a_{m-1}\, \cdots\, a_2\, a_1)$$

so its length is $2m - 1$ (we call m the *fan size*);

(iii) $S_k(m)$ is a Davenport Schinzel sequence of order 4.

The doubly inductive definition of $S_k(m)$ proceeds as follows.

I. $S_1(m) = (1\ 2\ \cdots\ m-1\ m\ m-1\ \cdots\ 2\ 1)$ for each $m \geq 1$.

II. $S_k(1)$ is the sequence $S_{k-1}(2^{k-1})$; each fan of length $2^k - 1$ in $S_{k-1}(2^{k-1})$ is regarded in $S_k(1)$ as $2^k - 1$ fans of size (and length) 1.

III. To obtain $S_k(m)$ for $k > 1$, $m > 1$, we proceed as follows.

(a) Construct $S' = S_k(m-1)$. S' has $F_k(m-1)$ fans, each of size $m-1$.

(b) Create $2F_{k-1}(F_k(m-1))$ distinct copies of S' (with pairwise disjoint sets of symbols). These copies have $2F_k(m-1) \cdot F_{k-1}(F_k(m-1)) = F_k(m)$ fans altogether.

(c) Construct $S^* = S_{k-1}(F_k(m-1))$. S^* has $F_{k-1}(F_k(m-1))$ fans, each of size $F_k(m-1)$. Duplicate the middle element of each fan of S^*. The total length of the modified S^* is $2F_k(m-1) \cdot F_{k-1}(F_k(m-1)) = F_k(m)$.

(d) For each $\beta \leq F_{k-1}(F_k(m-1))$, merge the β^{th} expanded fan of the modified S^* with the $(2\beta-1)^{th}$ and the $(2\beta)^{th}$ copies of S', by inserting the α^{th} element of the first half (resp. the second half) of the fan into the middle place of the α^{th} fan of the $(2\beta-1)^{th}$ (resp. the $(2\beta)^{th}$) copy of S', for each $\alpha \leq F_k(m-1)$, thereby duplicating the formerly middle element of each of these fans.

(e) $S_k(m)$ is just the concatenation of all these modified copies of S'.

Theorem 3.1 *$S_k(m)$ satisfies conditions (i)-(iii) stated above.*

Proof: By double induction on k and m. Clearly $S_1(m)$ satisfies these conditions for each $m \geq 1$. For arbitrary k, m, condition (i) is a direct consequence of the inductive construction and definition of $N_k(m)$.

As to condition (ii), the inductive construction and definition of $F_k(m)$ imply that $S_k(m)$ is the concatenation of $F_k(m)$ fans. That each fan consists of m distinct symbols and has the required form also follows from the inductive construction of the sequences.

As to condition (iii), we first observe that no pair of adjacent elements of $S_k(m)$ can be identical. Indeed, by the induction hypothesis this is the case for each copy of S' and for $S^\star$. The only duplications of adjacent elements which are effected by our construction is of the middle elements of all the fans of the copies of S' and of $S^\star$. However, in $S_k(m)$, an element of $S^\star$ is inserted between the two duplicated appearances of the middle element of each fan of any copy of S', and the two duplicated appearances of the middle element of a fan of $S^\star$ are inserted into two different fans in two different copies of S'. Thus $S_k(m)$ contains no pair of adjacent equal elements.

We also claim that $S_k(m)$ does not contain an alternation of the form

$$a \cdots b \cdots a \cdots b \cdots a \cdots b$$

for any pair of distinct symbols a and b. Indeed, by the induction hypothesis this holds if both a and b belong to $S^\star$ or if both belong to the same copy of S'. If a and b belong to two different copies of S' then these two copies are not interspersed at all in $S_k(m)$. The only remaining cases are when a belongs to $S^\star$ and b to some copy S'_β of S' or vice versa. In the first case only a single appearance of a (in the first or second half of the corresponding fan of $S^\star$) is inserted into S'_β, so the largest possible alternation between a and b in $S_k(m)$ is $a \cdots b \cdots a \cdots b \cdots a$. This same observation rules out the latter possibility (a belongs to S'_β and b to $S^\star$). Thus it follows by induction that $S_k(m)$ is a Davenport Schinzel sequence of order 4.

□

It remains to estimate the length $|S_k(m)|$ of $S_k(m)$ as a function of its number of symbols $N_k(m)$. Clearly

$$\frac{|S_k(m)|}{N_k(m)} = \frac{(2m-1)F_k(m)}{N_k(m)}.$$

To bound this from below, we will obtain an upper bound on $\dfrac{N_k(m)}{F_k(m)}$, as follows.

Theorem 3.2 $\dfrac{N_k(m)}{F_k(m)} \leq m \cdot D_k$, *where* $D_k = \prod_{j=1}^{k} c_j$ *for* $k \geq 1$ *and*

$$c_j = \frac{1}{2 - \frac{1}{2^{j-1}}} \qquad \text{for } j \geq 1.$$

Proof: For $k = 1$ we have $D_1 = c_1 = 1$ and $\dfrac{N_1(m)}{F_1(m)} = m = m \cdot D_1$, as required.

For $k > 1$ and $m = 1$ we have

$$\begin{aligned}
\frac{N_k(1)}{F_k(1)} &= \frac{N_{k-1}(2^{k-1})}{(2^k-1)F_{k-1}(2^{k-1})} \\
&\leq \frac{2^{k-1}}{2^k-1} \cdot D_{k-1} \\
&= c_k \cdot D_{k-1} = 1 \cdot D_k
\end{aligned}$$

as required.

For $k > 1$ and $m > 1$ we have

$$\begin{aligned}
\frac{N_k(m)}{F_k(m)} &= \frac{N_k(m-1)}{F_k(m-1)} + \frac{1}{2F_k(m-1)} \cdot \frac{N_{k-1}(F_k(m-1))}{F_{k-1}(F_k(m-1))} \\
&\leq (m-1) \cdot D_k + \frac{1}{2F_k(m-1)} \cdot F_k(m-1) \cdot D_{k-1} \\
&= (m-1)D_k + \frac{1}{2}D_{k-1} \\
&\leq m \cdot D_k \qquad \text{(because } \tfrac{1}{2} < c_k\text{)}.
\end{aligned}$$

□

Corollary 3.3

$$\frac{|S_k(m)|}{N_k(m)} \geq \frac{2m-1}{m} \cdot 2^k \cdot \prod_{j=1}^{\infty}(1-\frac{1}{2^j})$$

Proof: By Theorem 3.2,

$$\begin{aligned}\frac{|S_k(m)|}{N_k(m)} &\geq \frac{2m-1}{m \cdot D_k} = \frac{2m-1}{m} \cdot \prod_{j=1}^{k}(2(1-\frac{1}{2^j})) \\ &\geq \frac{2m-1}{m} \cdot 2^k \cdot \prod_{j=1}^{\infty}(1-\frac{1}{2^j})\end{aligned}$$

(the limit of the last infinite product is easily seen to be positive).

□

Theorem 3.4

$$\lambda_4(n) = \Omega(n \cdot 2^{\alpha(n)})$$

Proof: Put $\beta = \prod_{j=1}^{\infty}(1-\frac{1}{2^j})$. Clearly $0 < \beta < 1$. Theorem 3.2 and property (P.7) imply $N_k(1) \leq F_k(1) \leq A_k(5)$ for all $k \geq 3$. Hence for each $k \geq 5$ we have

$$n_k \equiv N_k(1) \leq A_k(k) = A(k)$$

so that $\alpha(n_k) \leq k$. On the other hand, the sequence $\{n_k\}_{k\geq 1}$ is easily seen to converge to infinity. Thus, for any given n, we find k such that

$$n_k \leq n < n_{k+1}$$

Assume with out loss of generality that $k \geq 4$. Put $t = \left\lfloor \frac{n}{n_k} \right\rfloor$ so that

$$t \cdot n_k \leq n < (t+1) \cdot n_k < 2t \cdot n_k.$$

Clearly,
$$\begin{aligned}\lambda_4(n) &\geq t\lambda_4(n_k) \geq t \cdot |S_k(1)| \\ &\geq \beta t \cdot n_k \cdot 2^k \qquad \text{(by Corollary 3.3)} \\ &> \frac{\beta}{2} n \cdot 2^k\end{aligned}$$

But $\alpha(n) \leq \alpha(n_{k+1}) \leq k+1$ so that $k \geq \alpha(n) - 1$, and we thus have

$$\lambda_4(n) \geq \frac{\beta}{4} n \cdot 2^{\alpha(n)}$$

for all $n \geq N_4(1)$. For smaller values of n we have $\alpha(n) \leq 5$, $\beta < \frac{3}{8}$, so we have to show that $\lambda_4(n) \geq 3n$, which is easily checked to hold for all $n \geq 3$. For $n = 1, 2$ the asserted inequality is trivial, thus we have for each $n \geq 1$

$$\lambda_4(n) \geq \frac{\beta}{4} n \cdot 2^{\alpha(n)} = \Omega(n \cdot 2^{\alpha(n)})$$

□

Corollary 3.5 $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$.

Proof: The above relation immediately follows from the results of Theorem 2.12 and 3.4.

□

4 The Upper Bounds for $\lambda_s(n)$:

In this section we extend the approach of section 2 to obtain improved upper bounds for $\lambda_s(n)$. In particular, we show that

$$\lambda_s(n) \leq \begin{cases} n \cdot 2^{(\alpha(n))^{\frac{s-2}{2}}+C_s(n)} & \text{if } s \text{ is even} \\ n \cdot 2^{(\alpha(n))^{\frac{s-3}{2}}\log(\alpha(n))+C_s(n)} & \text{if } s \text{ is odd} \end{cases} \tag{4.1}$$

where $C_s(n)$ satisfies the following bound

$$3+s \;\leq\; C_s(n) = \begin{cases} 6 & \text{if } s=3 \\ 11 & \text{if } s=4 \\ O\left((\alpha(n))^{\frac{s-4}{2}} \cdot \log(\alpha(n))\right) & \text{if } s>4 \text{ is even} \\ O\left((\alpha(n))^{\frac{s-3}{2}}\right) & \text{if } s>3 \text{ is odd} \end{cases} \tag{4.2}$$

A more precise definition of $C_s(n)$ is given in (4.4).

In [DS] , [At], it has been proved that $\lambda_s(n) \leq \frac{n(n-1)}{2} \cdot s + 1$. For $n \leq 4$ and $s \geq 3$ we can directly verify that

$$\lambda_s(n) \leq n \cdot 2^{(\alpha(n))^{\frac{s-3}{2}}+C_s(n)}.$$

For $4 < n \leq 16$ we have $\alpha(n) \geq 2$ and

$$\begin{aligned} \lambda_s(n) \;&\leq\; 8s \cdot n \quad = 2^{3+\log s} \cdot n \\ &\leq\; n \cdot 2^{2^{\frac{s-3}{2}}+3+s} \quad \leq n \cdot 2^{(\alpha(n))^{\frac{s-3}{2}}+C_s(n)}. \end{aligned}$$

Thus for $n \leq 16$, $\lambda_s(n)$ satisfies the desired inequality. Therefore, we restrict our attention to $n > 16$. It can be easily verified that the above inequality holds for $s = 3$ and $s = 4$. For $s = 3$, Hart and Sharir [HS] proved that

$$\begin{aligned} \lambda_3(n) \;&\leq\; 52 \cdot n\alpha(n) \\ &=\; n \cdot 2^{\log 52+\log(\alpha(n))} \\ &\leq\; n \cdot 2^{6+\log(\alpha(n))} \\ &=\; n \cdot 2^{\log(\alpha(n))+C_3(n)} \end{aligned}$$

For $s = 4$, the equation 2.4 actually gives, for $n > 16$

$$\begin{aligned} \lambda_4(n) \;&\leq\; 2^{11} \cdot n \cdot 2^{\alpha(n)} \\ &=\; n \cdot 2^{\alpha(n)+C_4(n)} \end{aligned}$$

For $s > 4$, we prove the desired upper bound (4.1) for $\lambda_s(n)$ by induction on s.

In this section, apart from the Ackermann's function, we need some more functions defined in terms of $\alpha(n)$. Let $\{\Gamma_s\}_{s>2}$ be a sequence of functions defined on $\mathcal{N}$ by:

$$\Gamma_s(n) = \begin{cases} (\alpha(n))^{\frac{s-2}{2}} & \text{if } s \text{ is even} \\ (\alpha(n))^{\frac{s-3}{2}} \cdot \log(\alpha(n)) & \text{if } s \text{ is odd} \end{cases} \tag{4.3}$$

Therefore, $\Gamma_2(n) = 1$, $\Gamma_3(n) = \log(\alpha(n))$ and for all $s \geq 4$, $\Gamma_s(n) = \Gamma_{s-2}(n) \cdot \alpha(n)$.

We define $\{C_s(n)\}_{s\geq 3}$ as follows:

$$C_s(n) = \sum_{i=2}^{s-1} a_i^s \cdot \Gamma_i(n) \tag{4.4}$$

where a_i^s is a constant depending on the value of i and s and is defined recursively as follows:

$$a_2^3 = 6 \qquad a_2^4 = 11 \qquad a_3^4 = 0$$

and for $s > 4$

$$a_i^s = \begin{cases} a_{s-3}^{s-2} + 1 & \text{if } i = s-1 \\ a_{i-2}^{s-2} + a_i^{s-1} & \text{if } 3 < i < s-1 \\ a_i^{s-1} & \text{if } i \leq 3 \end{cases} \tag{4.5}$$

and finally, let $\Pi_s(n)$ be

$$\Pi_s(n) = 2^{\Gamma_s(n)+C_s(n)} \qquad \text{for } s \geq 3 \tag{4.6}$$

Note that for each fixed n, $\{\Pi_s(n)\}_{s\geq 3}$ is increasing and for each fixed s, $\{\Pi_s(n)\}_{n\geq 1}$ is also increasing. From the definition of $\Pi_s(n)$ it follows that to prove the desired upper bound for $\lambda_s(n)$, we have to show that

$$\lambda_s(n) \leq n \cdot \Pi_s(n).$$

4.1 Upper Bounds For $\Psi_s(m, n)$:

In this subsection we establish an upper bound on the maximal length $\Psi_s(m, n)$ of an (n, s) of Davenport-Schinzel sequence composed of at most m 1-chains and having maximal length. The following lemma is a (somewhat modified) extension of Lemma 2.9.

Lemma 4.1 *Let $m, n \geq 1$ and $1 < b < m$ be integers. Then for any partitioning $m = \sum_{i=1}^{b} m_i$ with $m_1, \ldots, m_b \geq 1$ there exist integers $n^\star$, $n_1, n_2, \ldots, n_b \geq 0$ such that*

$$n^\star + \sum_{i=1}^{b} n_i = n$$

and

$$\Psi_s(m, n) \leq \Psi_s^{s-2}(b, n^\star) + 2 \cdot \Psi_{s-1}(m, n^\star) + 4m + \sum_{i=1}^{b} \Psi_s(m_i, n_i) \tag{4.7}$$

Proof: The proof given in Lemma 2.9 can be extended to handle the general case. Let U be a $DS(n, s)$-sequence consisting of at most m 1-chains $c_1, \ldots, c_m$ such that $|U| = \Psi_s(m, n)$. Partition the sequence into b layers (i.e. disjoint contiguous subsequences) $L_1, \ldots, L_b$ so that the layer L_i consists of m_i chains. Call a symbol a *internal* or *external* as in Lemma 2.9. Suppose there are n_i internal symbols in layer L_i, and $n^\star$ external symbols (thus $n^\star + \sum_{i=1}^{b} n_i = n$).

Using the same argument as in Lemma 2.9 we can show that the total contribution of internal symbols to $|U|$ is at most

$$m - b + \sum_{i=1}^{b} \Psi_s(m_i, n_i).$$

We bound the total number of occurrences of external symbols in three parts instead of two as in Lemma 2.9. For each layer L_i, call an external symbol a a *starting symbol* if its first (i.e. leftmost) occurrence is in L_i, an *ending symbol* if its last (i.e. rightmost) occurrence is in L_i, and a *middle symbol* if it is neither a starting nor an ending symbol. An external symbol appears as a starting symbol or an ending symbol exactly in one layer. First we estimate the total number of occurrences of middle symbols. For each layer L_i erase all internal symbols, starting symbols and ending symbols. Also erase each occurrence of a middle symbol which has become equal to the element immediately preceding it (there are at most $m_i - 1$ such erasures). Let us denote the resultant sequence by $L_i^\star$.

By generalizing the argument given in the proof of Lemma 2.9, it can be easily shown that $L_i^\star$ is a $DS(p_i, s-2)$ sequence. Thus the concatenation of all sequences $L_i^\star$, with the additional possible deletions of any first element of $L_i^\star$ which happens

to be equal to the last element of $L^\star_{i-1}$, is a $DS(n^\star, s)$ sequence composed of b $(s-2)$-chains, and therefore the contribution of the middle symbols is at most

$$\Psi_s^{s-2}(b, n^\star) + m$$

Now, consider the contribution of the starting external symbols. For each layer $L^\star_i$, erase all internal symbols, middle symbols and ending symbols and if necessary also erase each occurrence of a starting symbol if it is equal to the element immediately preceding it. The above process deletes at most $m_i - 1$ starting symbols. Let us denote the resultant sequence by $L^\#_i$. Let q_i be the number of distinct symbols in $L^\#_i$. We claim that $L^\#_i$ is a $DS(q_i, s-1)$ sequence. Indeed, if it were not the case, $L^\#_i$ would have contained an alternating sequence of the form

$$\underbrace{a \quad b \quad a \ldots a \quad b}_{s+1}$$

if s is odd and

$$\underbrace{a \quad b \quad a \ldots b \quad a}_{s+1}$$

if s is even. Since b and a are external symbols and their first appearance is in $L^\#_i$, they also appear in some layers after $L^\#_i$. But then U contains an alternation of a and b having length $s+2$, which is impossible. Hence $L^\#_i$ is a $DS(q_i, s-1)$ sequence consisting of m_i 1-chains, so its length is at most $\Psi_{s-1}(m_i, q_i)$. Summing over all the layers, the contribution of starting symbols is at most

$$m + \sum_{i=1}^{b} \Psi_{s-1}(m_i, q_i) \le m + \Psi_{s-1}(m, \sum_{i=1}^{b} q_i)$$

But an external symbol appears as a starting symbol only in one layer, therefore $\sum_{i=1}^{b} q_i = n^\star$. Hence the total contribution of starting symbols is bounded by

$$m + \Psi_{s-1}(m, n^\star)$$

Since the ending symbols are symmetric to the starting symbols, the same bound holds for the number of appearances of ending symbols also. Therefore the total contribution of the external symbols is bounded by

$$3m + 2 \cdot \Psi_{s-1}(m, n^\star) + \Psi_s^{s-2}(b, n^\star)$$

Thus we obtain the desired inequality

$$\Psi_s(m, n) \leq \Psi_s^{s-2}(b, n^*) + 2 \cdot \Psi_{s-1}(m, n^*) + 4m + \sum_{i=1}^{b} \Psi_s(m_i, n_i)$$

$\square$

Remark: Note that in the above proof, we estimate the contribution of external symbols in three parts instead of two as in Lemma 2.9. The reason is that while the treatment of starting and ending external symbols as a single case can be extended to *even* values of s, it fails for odd values, because the resulting sequence $L_i^{\#}$ might be of order s rather than $s-1$, e.g. if a is a starting symbol and b is an ending symbol, then it is possible that a and b have $s+1$ alternations in the layer L_i (starting with a and ending with b). That is why, in general, partitioning the external symbols into two parts is not enough. Also the extra overhead for even values of s is negligible.

The proof of our upper bound proceeds by induction on s. The base cases $s = 3$ and $s = 4$ have already been discussed above. Let $s > 4$ and suppose the upper bound holds for each $t < s$, i.e. $\lambda_t(n) \leq n \cdot \Pi_t(n)$. Before giving the solution of the equation 4.7, we bound $\Psi_s^t(m, n)$ in terms of $\Psi_s(m, n)$.

Lemma 4.2 *Let $m, n \geq 1$ and $3 \leq t < s$ be integers; then*

$$\Psi_s^t(m, n) \leq \Pi_t(n) \cdot \Psi_s(m, n) + (m-1) \cdot \Pi_t(n)$$

Proof: This lemma is basically a generalization of Lemma 2.2. Let U be a $DS(n, s)$ sequence composed of m t-chains and having maximal length. Replace each chain c_i of U by the 1-chain c_i' composed of the same symbols in the order of their leftmost appearance in c_i. Since by the inductive hypothesis $\lambda_t(n) \leq n \cdot \Pi_t(n)$, we have $|c_i| \leq |c_i'| \cdot \Pi_t(n)$. Construct another sequence U' by concatenating all the 1-chains c_i' and erasing each first symbol of c_i' which is equal to its immediately preceding element. It is clear that U' is a $DS(n, s)$ sequence composed of at most m 1-chains and its length is at least $\sum_{i=1}^{m} |c_i'| - (m-1)$. Therefore

$$\Psi_s(m, n) \geq \frac{1}{\Pi_t(n)} \cdot \sum_{i=1}^{m} |c_i| - (m-1)$$

But, $\Psi_s^t(m, n) = \sum_{i=1}^{m} |c_i|$. Thus

$$\Psi_s^t(m, n) \leq \Pi_t(n) \cdot \Psi_s(m, n) + (m-1) \cdot \Pi_t(n)$$

□

Corollary 4.3

$$\lambda_s(n) \leq \Psi_s(2n-1, n) \cdot \Pi_{s-2}(n) + (2n-2) \cdot \Pi_{s-2}(n) \tag{4.8}$$

Proof: The proof directly follows from Lemma 2.1 and 4.2.

□

Lemma 4.4 *Let $m, n \geq 1$, and $k \geq 2$. Then*

$$\Psi_s(m, n) \leq \mathcal{F}_k(n) \cdot m \cdot \alpha_k(m) + \mathcal{G}_k(n) \cdot n \tag{4.9}$$

where $\mathcal{F}_k(n)$ and $\mathcal{G}_k(n)$ are defined recursively as follows:

$$\begin{aligned} \mathcal{F}_2(n) &= 4 \\ \mathcal{F}_k(n) &= 2\Pi_{s-2}(n) \cdot \mathcal{F}_{k-1}(n) + (\Pi_{s-2}(n) + 4) \qquad (4.10) \\ \mathcal{G}_2(n) &= 5\Pi_{s-1}(n) \\ \mathcal{G}_k(n) &= \Pi_{s-2}(n) \cdot \mathcal{G}_{k-1}(n) + 2\Pi_{s-1}(n) \qquad (4.11) \end{aligned}$$

Proof: $\Psi_s(m, n) \leq \lambda_s(n)$, therefore $\Psi_{s-1}(m, n^*) \leq n^* \cdot \Pi_{s-1}(n^*)$. If we replace $\Psi_{s-1}(m, n^*)$ by this bound in the equation (4.7) and also replace $\Psi_s^{s-2}(b, n^*)$ by the right hand side of the bound of lemma 4.2, we get:

$$\begin{aligned} \Psi_s(m, n) \leq\ & \Pi_{s-2}(n^*) \cdot \Psi_s(b, n^*) + (b-1) \cdot \Pi_{s-2}(n^*) + 4m + \\ & 2n^* \cdot \Pi_{s-1}(n^*) + \sum_{i=1}^{b} \Psi_s(m_i, n_i) \end{aligned} \tag{4.12}$$

We use the equation (4.12) repeatedly to obtain the desired bound for $k = 2, 3, \cdots$. At each step we choose b appropriately and estimate $\Psi_s(b, n^*)$ using a technique similar to Lemma 2.10.

We proceed by double induction on k and m. Initially $k = 2$, and $\alpha_k(m) = \log m$ for $m \geq 1$. For $k = 2$ choose $b = 2$, $m_1 = \lfloor \frac{m}{2} \rfloor$ and $m_2 = \lceil \frac{m}{2} \rceil$ in the equation (4.12); $\Psi_s(b, n^\star) = \Psi_s(2, n^\star) = 2n^\star$ for all $n^\star$ so (4.12) becomes

$$\begin{aligned}\Psi_s(m, n) &\leq \Psi_s(\lfloor \tfrac{m}{2} \rfloor, n_1) + \Psi_s(\lceil \tfrac{m}{2} \rceil, n_2) + \Pi_{s-2}(n^\star) + 4m + \\ &\quad 2n^\star \cdot (\Pi_{s-1}(n^\star) + \Pi_{s-2}(n^\star)) \\ &\leq \Psi_s(\lfloor \tfrac{m}{2} \rfloor, n_1) + \Psi_s(\lceil \tfrac{m}{2} \rceil, n_2) + 4m + \\ &\quad n^\star \cdot (2\Pi_{s-1}(n^\star) + 3\Pi_{s-2}(n^\star))\end{aligned}$$

where $n = n_1 + n_2 + n^\star$. The solution of this recurrence relation is easily seen to be

$$\Psi_s(m, n) \leq 4m \cdot \lceil \log m \rceil + n(2\Pi_{s-1}(n) + 3\Pi_{s-2}(n))$$

Since $\Pi_{s-1}(n) > \Pi_{s-2}(n)$, for $k = 2$ we have

$$\begin{aligned}\Psi_s(m, n) &\leq 4m \cdot \lceil \log m \rceil + 5n \cdot \Pi_{s-1}(n) \\ &= m \cdot \mathcal{F}_2(n) \cdot \alpha_2(m) + \mathcal{G}_2(n) \cdot n\end{aligned}$$

as asserted. For $k > 2$ and $m \leq 16$ the inequality (4.9) obviously holds as $\Psi_s(m, n) \leq 16n$ and the right hand side of (4.9) is $\geq 16n$. Now suppose that $k > 2$ and $m > 16$ and that the inductive hypothesis holds for all $k' < k$ and $m' \geq 1$ and for $k' = k$ and for all $m' < m$. Choose $t = \left\lceil \frac{\alpha_{k-1}(m)}{2} \right\rceil$, and $b = \left\lfloor \frac{m}{t} \right\rfloor$. For $k > 2$, $\alpha_{k-1}(m) \leq \lceil \log m \rceil$; thus

$$t \leq \left\lceil \frac{\lceil \log m \rceil}{2} \right\rceil < m - 1.$$

Suppose $m = b \cdot t + r$, then any $DS(n, s)$ sequence U composed of m 1-chains can be decomposed into b layers, $L_1, \ldots, L_b$ containing $m_1, \ldots, m_b$ 1-chains, so that $m_i = t + 1$ for $i \leq r$ and $m_i = t$ for the remaining layers; therefore $m_i \leq t + 1 < m$ for all i. By induction hypothesis (for $k - 1$ and b) we have

$$\Psi_s(b, n^\star) \leq \mathcal{F}_{k-1}(n^\star) \cdot b \cdot \alpha_{k-1}(b) + \mathcal{G}_{k-1}(n^\star) \cdot n^\star$$

But

$$\begin{aligned}b &\leq \frac{m}{t} = \frac{m}{\left\lceil \frac{\alpha_{k-1}(m)}{2} \right\rceil} \\ &\leq \frac{2m}{\alpha_{k-1}(m)}\end{aligned}$$

Clearly $b \le m$, therefore $\alpha_{k-1}(b) \le \alpha_{k-1}(m)$ and we have

$$\Psi_s(b,\, n^\star) \;\le\; \mathcal{F}_{k-1}(n^\star) \cdot 2m + \mathcal{G}_{k-1}(n^\star) \cdot n^\star$$

Since each $m_i < m$, by inductive hypothesis:

$$\sum_{i=1}^{b} \Psi_s(m_i,\, n_i) \quad\le\quad \sum_{i=1}^{b} (\mathcal{F}_k(n_i) \cdot m_i \cdot \alpha_k(m_i) + \mathcal{G}_k(n_i) \cdot n_i)$$

The value of $\alpha_k(m_i)$ can be estimated as follows:

$$\begin{aligned} \alpha_k(m_i) &\le \alpha_k(t+1) \\ &= \alpha_k\left(\left\lceil \frac{\alpha_{k-1}(m)}{2} \right\rceil + 1\right) \end{aligned}$$

Now for $m \ge 16$, $\alpha_{k-1}(m) \ge 3$ and then it is easy to check that

$$\left\lceil \frac{\alpha_{k-1}(m)}{2} \right\rceil + 1 \;\le \alpha_{k-1}(m).$$

Thus

$$\begin{aligned} \alpha_k(m_i) &\le \alpha_k(\alpha_{k-1}(m)) \\ &= \alpha_k(m) - 1 \end{aligned}$$

which implies

$$\begin{aligned} \sum_{i=1}^{b} \Psi_s(m_i,\, n_i) &\le \sum_{i=1}^{b} [\mathcal{F}_k(n_i) \cdot m_i \cdot (\alpha_k(m) - 1) + \mathcal{G}_k(n_i) \cdot n_i] \\ &\le m \cdot \mathcal{F}_k(n) \cdot (\alpha_k(m) - 1) + \mathcal{G}_k(n) \cdot \sum_{i=1}^{b} n_i \end{aligned}$$

If we substitute these values of $\Psi_s(b,\, n^\star)$ and $\sum_i^b \Psi_s(m_i,\, n_i)$ in (4.12) and using the fact that $b \le m$, we get

$$\begin{aligned} \Psi_s(m,\, n) \;\le\; & \Pi_{s-2}(n) \cdot [\mathcal{F}_{k-1}(n) \cdot 2m + \mathcal{G}_{k-1}(n) \cdot n^\star] + \\ & (\Pi_{s-2}(n) + 4) \cdot m + 2n^\star \cdot \Pi_{s-1}(n) + \\ & \mathcal{F}_k(n) \cdot m(\alpha_k(m) - 1) + \mathcal{G}_k(n) \cdot \sum_{i=1}^{b} n_i \end{aligned}$$

$$\begin{aligned}
&\leq [2\Pi_{s-2}(n) \cdot \mathcal{F}_{k-1}(n) + (\Pi_{s-2}(n) + 4)] \cdot m + \\
&\quad [\Pi_{s-2}(n) \cdot \mathcal{G}_{k-1}(n) + 2\Pi_{s-1}(n)] \cdot n^* + \\
&\quad \mathcal{F}_k(n) \cdot m(\alpha_k(m) - 1) + \mathcal{G}_k(n) \cdot \sum_{i=1}^{b} n_i
\end{aligned}$$

which by definition of $\mathcal{F}_k(n)$ and $\mathcal{G}_k(n)$ is

$$\begin{aligned}
\Psi_s(m, n) &\leq \mathcal{F}_k(n) \cdot m\alpha_k(m) + \mathcal{G}_k(n) \cdot (n^* + \sum_{i=1}^{b} n_i) \\
&= \mathcal{F}_k(n) \cdot m\alpha_k(m) + \mathcal{G}_k(n) \cdot n
\end{aligned}$$

Hence, the lemma is true.

□

Lemma 4.5 *For $k \geq 2$ $\mathcal{F}_k(n)$ and $\mathcal{G}_k(n)$ satisfy the following inequalities:*

$$\mathcal{F}_k(n) \leq 5 \cdot (2\Pi_{s-2}(n))^{k-2} \tag{4.13}$$

$$\mathcal{G}_k(n) \leq 6\Pi_{s-1}(n) \cdot (\Pi_{s-2}(n))^{k-2} \tag{4.14}$$

Proof: It is not difficult to see that a recurrence relation of the form

$$\begin{aligned}
T(2) &= c \\
T(k) &= aT(k-1) + b
\end{aligned}$$

has the following solution

$$T(k) = c \cdot a^{k-2} + \frac{a^{k-2} - 1}{a - 1} \cdot b$$

The recursive definition of $\mathcal{F}_k(n)$ given in (4.10) has the same form with $a = 2\Pi_{s-2}(n)$, $b = (\Pi_{s-2}(n) + 4)$ and $c = 4$. Therefore

$$\mathcal{F}_k(n) \leq 4 \cdot (2\Pi_{s-2}(n))^{k-2} + \frac{\left[(2\Pi_{s-2}(n))^{k-2} - 1\right]}{[2\Pi_{s-2}(n) - 1]} \cdot (\Pi_{s-2}(n) + 4)$$

But for $x > 5$ $\frac{x+4}{2x-1} < 1$. Since $\Pi_{s-2}(n) > 5$, we get

$$\mathcal{F}_k(n) \leq 5 \cdot (2\Pi_{s-2}(n))^{k-2}$$

Similarly, the recursive definition of $\mathcal{G}_k(n)$ given in (4.11) also has the same form with $a = \Pi_{s-2}(n)$, $b = 2\Pi_{s-1}(n)$ and $c = 5\Pi_{s-1}(n)$. Hence

$$\mathcal{G}_k(n) \leq 5\Pi_{s-1}(n) \cdot (\Pi_{s-2}(n))^{k-2} + \frac{\left[(\Pi_{s-2}(n))^{k-2} - 1\right]}{[\Pi_{s-2}(n) - 1]} \cdot 2\Pi_{s-1}(n)$$

Since $\Pi_{s-2}(n) > 4$, $\frac{2}{[\Pi_{s-2}(n)-1]} < 1$, we get

$$\mathcal{G}_k(n) \leq 6\Pi_{s-1}(n) \cdot (\Pi_{s-2}(n))^{k-2}$$

□

Theorem 4.6 *For $s \geq 2$, $n \geq 1$*

$$\lambda_s(n) \leq n \cdot \Pi_s(n)$$

Proof: If we substitute $k = \alpha(n)$ in (4.9) we get

$$\Psi_s(m, n) \leq \mathcal{F}_{\alpha(n)}(n) \cdot m\alpha_{\alpha(n)}(m) + \mathcal{G}_{\alpha(n)}(n) \cdot n$$

Now we can use the corollary 4.3 to bound $\lambda_s(n)$. Substitute the above value of $\Psi_s(m, n)$ in the equation (4.8):

$$\begin{aligned}\lambda_s(n) \leq\ & \Pi_{s-2}(n) \cdot \mathcal{F}_{\alpha(n)}(n) \cdot (2n-1) \cdot \alpha_{\alpha(n)}(2n-1) + \\ & \Pi_{s-2}(n) \cdot \mathcal{G}_{\alpha(n)}(n) \cdot n + (2n-2) \cdot \Pi_{s-2}(n)\end{aligned}$$

For $k \geq 2$, $\alpha_k(2n) \leq (\alpha_k(n) + 1)$ and $\alpha_{\alpha(n)}(n) \leq \alpha(n)$, so we have

$$\begin{aligned}\lambda_s(n) &\leq n \cdot \Pi_{s-2}(n) \cdot \left[2\mathcal{F}_{\alpha(n)}(n) \cdot (\alpha(n) + 1) + \mathcal{G}_{\alpha(n)}(n) + 2\right] \\ &\leq n \cdot \Pi_{s-2}(n) \cdot \left[4\mathcal{F}_{\alpha(n)}(n) \cdot \alpha(n) + \mathcal{G}_{\alpha(n)}(n)\right]\end{aligned}$$

After substituting the values of $\mathcal{F}_{\alpha(n)}(n)$ and $\mathcal{G}_{\alpha(n)}(n)$ from (4.13) and (4.14), the above inequality becomes

$$\begin{aligned}\lambda_s(n) &\leq n \cdot \Pi_{s-2}(n) \cdot \Big[4 \cdot 5(2\Pi_{s-2}(n))^{\alpha(n)-2} \cdot \alpha(n) + \\ &\qquad 6(\Pi_{s-2}(n))^{\alpha(n)-2} \cdot \Pi_{s-1}(n)\Big] \\ &\leq n \cdot (\Pi_{s-2}(n))^{\alpha(n)-1} \cdot \Big[5\alpha(n) \cdot 2^{\alpha(n)} + 6\Pi_{s-1}(n)\Big]\end{aligned}$$

Since for all $s > 4$,

$$\Pi_{s-1}(n) \geq \Pi_4(n) = 2^{\alpha(n)+8},$$

we get

$$\lambda_s(n) \leq n \cdot (\Pi_{s-2}(n))^{\alpha(n)} \cdot \Pi_{s-1}(n) \cdot \frac{\left[\frac{5\alpha(n)}{2^{k}} + 6\right]}{\Pi_{s-2}(n)}$$

But for $n > 16$, $\Pi_{s-2}(n) \geq \Pi_3(n) = 64\alpha(n)$, therefore

$$\lambda_s(n) \leq n \cdot (\Pi_{s-2}(n))^{\alpha(n)} \cdot \Pi_{s-1}(n) \qquad = n \cdot 2^{\vartheta}$$

Putting the value of Π_{s-1} and Π_{s-2} from (4.6) we get

$$\begin{aligned}\vartheta &= \Gamma_{s-2}(n) \cdot \alpha(n) + \sum_{i=2}^{s-3} a_i^{s-2} \cdot \Gamma_i(n) \cdot \alpha(n) + \\ &\quad \Gamma_{s-1}(n) + \sum_{i=2}^{s-2} a_i^{s-1} \cdot \Gamma_i(n)\end{aligned}$$

But by definition, $\Gamma_i(n) \cdot \alpha(n) = \Gamma_{i+2}(n)$, therefore

$$\begin{aligned}\vartheta &= \Gamma_s(n) + \sum_{i=2}^{s-3} a_i^{s-2} \cdot \Gamma_{i+2}(n) + \Gamma_{s-1}(n) + \sum_{i=2}^{s-2} a_i^{s-1} \cdot \Gamma_i(n) \\ &= \Gamma_s(n) + (1 + a_{s-3}^{s-2}) \cdot \Gamma_{s-1}(n) + \sum_{i=4}^{s-2}(a_{i-2}^{s-2} + a_i^{s-1}) \cdot \Gamma_i(n) + \\ &\quad a_3^{s-1} \cdot \Gamma_3(n) + a_2^{s-1} \cdot \Gamma_2(n) \\ &= \Gamma_s(n) + \sum_{i=2}^{s-1} a_i^s \cdot \Gamma_i(n) \\ &= \Gamma_s(n) + C_s(n)\end{aligned}$$

Thus, we get the desired upper bound for $\lambda_s(n)$.

□

Corollary 4.7 *For $s \geq 2$ and for sufficiently large n*

$$\lambda_s(n) = n \cdot 2^{\Gamma_s(n)\cdot(1+o(1))}$$

Proof: We have already shown that the above equality holds for $s \leq 4$. Therefore, we assume that $s > 4$. By the definition of Γ_s, for all $i < s$

$$\lim_{n\to\infty} \frac{\Gamma_i(n)}{\Gamma_s(n)} = 0$$

Thus

$$\lim_{n\to\infty} \frac{C_s(n)}{\Gamma_s(n)} = 0$$

so that

$$\begin{aligned}\lambda_s(n) &\leq n \cdot 2^{\Gamma_s(n)\cdot\left(1+\frac{C_s(n)}{\Gamma_s(n)}\right)} \\ &= n \cdot 2^{\Gamma_s(n)(1+o(1))}\end{aligned}$$

which completes the proof.

□

Remarks:

(i) The above proof of the upper bound is similar to the one given by Sharir in [Sh1]. The main difference between the two proofs is that he estimated the contribution of the external symbols by dividing them into two parts: *starting symbols* and *non-starting symbols* while we divide them into three parts *starting*, *middle* and *ending* which allows us to write $\Psi_s(m, n)$ in terms of $\Psi_{s-2}(b, n^\star)$ instead of $\Psi_{s-1}(b, n^\star)$. Since we go two steps down at a time instead of one, we get a new $\alpha(n)$ term in the exponent only after increasing the value of s by 2 instead of every step. Moreover, in our bounds, we do not have any $\log(\alpha(n))$ term for even values of s, i.e. the base of the exponent in this case is 2, not $\alpha(n)$.

(ii) Note that in equation 4.12 we approximate $\Psi_{s-1}(m, n)$ to $\lambda_{s-1}(n)$ instead of substituting the bound achieved from (4.12) inductively as we do in the case of $s = 4$. If we substitute $\Psi_{s-1}(m, n)$ by (4.12) instead of approximating it to $\lambda_{s-1}(n)$, we can improve the upper bound for $\lambda_s(n)$ a little bit by optimizing the polynomial $C_s(n)$, however it does not affect the leading term and also as we will see in the next section, even then the bounds we obtain still do not match our lower bounds. Moreover, the proof also becomes much more complicated.

5 The Lower Bounds for $\lambda_s(n)$:

In this section we establish the lower bounds for $\lambda_s(n)$. We show that for $n \geq A(7)$ and even $s \geq 6$,

$$\lambda_s(n) \geq n \cdot 2^{K_s \alpha(n)^{\frac{s-2}{2}} + Q_s(n)}$$

where $K_s = \frac{1}{\frac{s-2}{2}!}$ and Q_s is a polynomial in $\alpha(n)$ of degree at most $\frac{s-4}{2}$ defined later in this section. These bounds improve significantly the previous lower bounds given by Sharir [Sh2] and almost match the upper bounds given in the previous section for even values of s.

The proof of this bound is quite similar to the proof of the lower bound for $s = 4$, only it is more complicated. Before we give the proof, we will need to define several functions which behave similarly to the Ackermann function, and prove certain properties about them. We will then define a collection of Davenport Schinzel sequences $S_k^s(m)$ of order s that realize our lower bounds.

5.1 The Functions $\mathbf{F_k^s(m)}$, $\mathbf{N_k^s(m)}$, $\mathbf{F_k^\omega(m)}$ and their properties

For the lower bounds we will need two classes of functions that grow faster than Ackermann's functions though roughly at the same rate. These functions, $F_k^s(m)$ and $N_k^s(m)$, are defined for integral $k \geq 1$, integral $m \geq 1$ and even $s \geq 2$. $N_k^s(m)$ gives the number of symbols composing the sequence $S_k^s(m)$, and $F_k^s(m)$ gives the number of fans in $S_k^s(m)$ (see below for more details). These functions are defined inductively by the following equations.

$$\begin{array}{lcll}
F_1^s(m) & = & 1 & m \geq 1,\ s \geq 2 \\
F_k^2(m) & = & 1 & m \geq 1,\ k \geq 1 \\
F_k^s(1) & = & (2^k - 1) \cdot F_{k-1}^{s-2}(2^{k-1}) \cdot F_{k-1}^s(N_{k-1}^{s-2}(2^{k-1})) & k \geq 2,\ s \geq 4 \\
F_k^s(m) & = & 2F_k^s(m-1) \cdot F_{k-1}^{s-2}(F_k^s(m-1)) \cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1))) & m \geq 2, k \geq 2, s \geq 4
\end{array}$$

$$
\begin{aligned}
N_1^s(m) &= m && m \geq 1,\ s \geq 2 \\
N_k^2(m) &= m && m \geq 1,\ k \geq 1 \\
N_k^s(1) &= N_{k-1}^s(N_{k-1}^{s-2}(2^{k-1})) && k \geq 2,\ s \geq 4 \\
N_k^s(m) &= N_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1)))+ \\
&\quad 2N_k^s(m-1) \cdot F_{k-1}^{s-2}(F_k^s(m-1)) \cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1))) && m \geq 2, k \geq 2, s \geq 4
\end{aligned}
$$

For $s = 4$, these formulas define the functions $F_k(m)$ and $N_k(m)$ that we used in the lower bounds for $\lambda_4(n)$.

We will now state several facts about the functions $F_k^s(m)$ and $N_k^s(m)$. Their proofs are given in Appendix 3. Notice that it is clear from the definitions that these functions are always positive integers.

Fact 5.1: For $m \geq 2$, $F_k^s(m) \geq F_k^s(m-1)$.

Fact 5.2: For $m \geq 2$, $N_k^s(m) \geq N_k^s(m-1)$.

These facts are trivially true when $k = 1$ or $s = 2$. For $k > 1$ we see that $F_k^s(m) \geq 2F_k^s(m-1)$ and $N_k^s(m) \geq 2N_k^s(m-1)$.

Recall the product D_k we used in the lower bound for $s = 4$. We will also need it in this bound, as well as some of its properties. The definition of D_k was

$$D_k = \prod_{j=1}^{k} \frac{2^{j-1}}{2^j - 1}.$$

We need

Fact 5.3: For $k \geq 2$, $D_k \leq 2^{-(k-2)}$.

Another function that we need is $P(k,s)$, defined on positive integers k and even positive integers s, as follows

$$P(k,s) = \sum_{i=1}^{\frac{s}{2}-1} \binom{k-2}{i}$$

where we define the binomial coefficient $\binom{a}{b}$ to be 0 if $a < b$.

Fact 5.4: For $k \geq 2$,

$$P(k, s) = \sum_{i=1}^{k-1} P(i, s-2) + k - 2.$$

We now show

Lemma 5.1

$$\frac{N_k^s(m)}{F_k^s(m)} \leq m \cdot 2^{-P(k,s)} \tag{5.1}$$

Proof: We will actually show that

$$\frac{N_k^s(m)}{F_k^s(m)} \leq m \cdot D_k \cdot 2^{-\sum_{i=1}^{k-1} P(i,s-2)}, \tag{5.2}$$

Facts 5.3 and 5.4 above show that this implies Lemma 5.1. The proof of this lemma will be by induction. During the induction, we use both this inequality and the one stated in the lemma for *smaller* values of s, k and m.

We prove that the inequality holds for m, k, and s, assuming that it holds for m', k' and s' whenever $s' < s$, or $k' < k$ and $s' = s$, or $m' < m$, $k' = k$ and $s' = s$.

Case 1: $s = 4$. In this case $P(i, s-2) = 0$ for $i \geq 1$, so we have to show

$$\frac{N_k^4(m)}{F_k^4(m)} \leq m \cdot D_k$$

which is what we have shown in Theorem 3.2.

Case 2: $k = 1$. In this case

$$\begin{aligned} \frac{N_1^s(m)}{F_1^s(m)} &= m = m \cdot 1 \\ &= m \cdot D_1 = m \cdot 2^{-P(1,s-2)} \end{aligned}$$

Case 3: $m = 1$. We have

$$\frac{N_k^s(1)}{F_k^s(1)} = \frac{N_{k-1}^s(N_{k-1}^{s-2}(2^{k-1}))}{(2^k-1) \cdot F_{k-1}^{s-2}(2^{k-1}) \cdot F_{k-1}^s(N_{k-1}^{s-2}(2^{k-1}))}$$

$$
\begin{aligned}
&\le \frac{N_{k-1}^{s-2}(2^{k-1})}{(2^k-1)\cdot F_{k-1}^{s-2}(2^{k-1})} \cdot D_{k-1} \cdot 2^{-\sum_{i=1}^{k-2} P(i,s-2)} \\
&\quad \text{(Using equation 5.2)} \\
&\le \frac{2^{k-1}}{2^k-1} \cdot D_{k-1} \cdot 2^{-P(k-1,s-2)} \cdot 2^{-\sum_{i=1}^{k-2} P(i,s-2)} \\
&\quad \text{(Using equation 5.1)} \\
&= D_k \cdot 2^{-\sum_{i=1}^{k-1} P(i,s-2)}.
\end{aligned}
$$

Case 4: $m > 1$.

We have

$$
\begin{aligned}
\frac{N_k^s(m)}{F_k^s(m)} &= \frac{N_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1)))}{2F_k^s(m-1)\cdot F_{k-1}^{s-2}(F_k^s(m-1))\cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1)))} + \\
&\quad \frac{N_k^s(m-1)}{F_k^s(m-1)} \\
&\le \frac{N_{k-1}^{s-2}(F_k^s(m-1))}{2F_k^s(m-1)\cdot F_{k-1}^{s-2}(F_k^s(m-1))} \cdot D_{k-1} \cdot 2^{-\sum_{i=1}^{k-2} P(i,s-2)} + \\
&\quad (m-1)\cdot D_k \cdot 2^{-\sum_{i=1}^{k-1} P(i,s-2)} \\
&\quad \text{(Using equation 5.2)} \\
&\le \frac{D_{k-1}}{2} \cdot 2^{-P(k-1,s-2)} \cdot 2^{-\sum_{i=1}^{k-2} P(i,s-2)} + \\
&\quad (m-1)\cdot D_k \cdot 2^{-\sum_{i=1}^{k-1} P(i,s-2)} \\
&\quad \text{(Using equation 5.1)} \\
&\le m \cdot D_k \cdot 2^{-\sum_{i=1}^{k-1} P(i,s-2)}
\end{aligned}
$$

since $\frac{D_{k-1}}{2} < D_k$.

□

We must still relate the functions F_k^s to the Ackermann's function. We will do this by using limit functions $F_k^\omega(m)$ such that $F_k^\omega(m) \ge F_k^s(m)$ for all s. We define

the limit function $F_k^\omega(m)$ by

$$\begin{array}{lll} F_1^\omega(m) & = & 1 \qquad m \geq 1 \\ F_k^\omega(1) & = & (2^k - 1) \cdot F_{k-1}^\omega(2^{k-1}) \cdot F_{k-1}^\omega(2^{k-1} \cdot F_{k-1}^\omega(2^{k-1})) \qquad k \geq 2 \\ F_k^\omega(m) & = & 2 \cdot F_k^\omega(m-1) \cdot F_{k-1}^\omega(F_k^\omega(m-1)) \cdot F_{k-1}^\omega(F_k^\omega(m-1) \cdot F_{k-1}^\omega(F_k^\omega(m-1))) \\ & & m \geq 2,\ k \geq 2 \end{array}$$

We will now show

Lemma 5.2 *For all s, $F_k^\omega(m) \geq F_k^s(m)$.*

Proof: See Appendix 3 for the proof.

□

We will now need to prove various facts about the functions F^ω.

Fact 5.5: $F_2^\omega(m) = 3 \cdot 2^{m-1}$.
This follows from the definitions: substituting $F_1^\omega(m) = 1$ in the recursions, we get $F_2^\omega(m) = 2 \cdot F_2^\omega(m-1)$ and $F_2^\omega(1) = 3$.

Fact 5.6: For $k \geq 2$, $2^a \cdot F_k^\omega(m) \leq F_k^\omega(m+a)$.
This follows from $F_k^\omega(m) \geq 2 \cdot F_k^\omega(m-1)$ which is immediate from the definition of F_k^ω.
Since $2^a > a$ for $a \geq 1$, Fact 5.6 implies

Fact 5.7: For $k \geq 2$, $a \cdot F_k^\omega(m) \leq F_k^\omega(m+a)$.

Finally, we show that

Fact 5.8: For $k \geq 2$, $A_k(m+1) \geq 2A_k(m)$.
(see Appendix 3 for the proof).

We are now ready to prove

Lemma 5.3 $F_k^\omega(m) \leq A_k(7m)$

Proof: See Appendix 3 for the proof.

□

5.2 The Sequences $S_k^s(m)$:

We will now define the Davenport Schinzel sequences of order s that we will use to prove our lower bound. The sequences of order s will be indexed by two variables, k and m, and called $S_k^s(m)$. The sequence $S_k^s(m)$ will be composed of $N_k^s(m)$ symbols. As in the case $s = 4$, the sequence $S_k^s(m)$ will be a concatenation of $F_k^s(m)$ fans of size m, where a fan of size m is composed of m distinct symbols $a_1, a_2, \cdots, a_m$ and has the form $(a_1 a_2 \cdots a_{m-1} a_m a_{m-1} \cdots a_2 a_1)$, so its length is $2m - 1$. In our construction, we will be replacing fans in certain subsequences by Davenport Schinzel sequences of order $s - 2$. When we replace a fan by a sequence, the sequence will contain the same symbols as in the replaced fan, and the first appearance of these symbols in the sequence will be in the same order as it was in the fan.

We will define $S_k^s(m)$ for even $s \geq 2$, and integral $k \geq 1$, $m \geq 1$. The definition of $S_k^s(m)$ we proceeds inductively as follows

I. $S_1^s(m) = (1\ 2\ \cdots\ m-1\ m\ m-1\ \cdots\ 2\ 1)$ for $s \geq 2$, and $m \geq 1$.

II. $S_k^2(m) = (1\ 2\ \cdots\ m-1\ m\ m-1\ \cdots\ 2\ 1)$ for $k, m \geq 1$.

III. To obtain $S_k^s(1)$ for $k > 1$, $s > 2$ proceed as follows:

(a) Construct the sequence $S' = S_{k-1}^s(N_{k-1}^{s-2}(2^{k-1}))$. S' has $F_{k-1}^s(N_{k-1}^{s-2}(2^{k-1}))$ fans, each of size $N_{k-1}^{s-2}(2^{k-1})$.

(b) Replace each fan of S' by the sequence $S_{k-1}^{s-2}(2^{k-1})$ using the same set of $N_{k-1}^{s-2}(2^{k-1})$ symbols as the replaced fan, with the first appearance of symbols in the same order.

(c) Regard each element of the resulting sequence as its own singleton fan.

IV. To obtain $S_k^s(m)$, for $k > 1$, $s > 2$, $m > 1$, we proceed as follows:

(a) First construct the sequence $S_k^s(m-1)$. It has $F_k^s(m-1)$ fans, each of size $m - 1$.

(b) Create $2 \cdot F_{k-1}^{s-2}(F_k^s(m-1)) \cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1)))$ distinct copies of it, having pairwise disjoint sets of symbols. Duplicate the middle element

of each fan in these copies of $S_k^s(m-1)$ and concatenate all these copies into a long sequence. Call this sequence S'. These copies have

$$2 \cdot F_k^s(m-1) \cdot F_{k-1}^{s-2}(F_k^s(m-1)) \cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1))) = F_k^s(m)$$

fans altogether.

(c) Now construct the sequence $S_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1)))$. It has

$$F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1)))$$

fans, each of size $N_{k-1}^{s-2}(F_k^s(m-1))$.

(d) Replace each of its fan of it by the sequence $S_{k-1}^{s-2}(F_k^s(m-1))$ using the same set of $N_{k-1}^{s-2}(F_k^s(m-1))$ symbols as in the replaced fan, making their first appearance in the same order. Duplicate the middle element of each fan of this sequence; these fans come from the sequences $S_{k-1}^{s-2}(F_k^s(m-1))$ and thus have size $F_k^s(m-1)$. Call this sequence $S^\star$.

(e) Notice that the sequence $S^\star$ has

$$2 \cdot F_k^s(m-1) \cdot F_{k-1}^{s-2}(F_k^s(m-1)) \cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1))) = F_k(m)$$

elements, which is the same as the number of fans in S'. To obtain $S_k^s(m)$, insert the sequence $S^\star$ into the sequence S', with each element of $S^\star$ going into the middle of a corresponding fan of S'. The fans of $S_k^s(m)$ are the fans of S', with the extra symbol from $S^\star$ added in the middle.

We will show that the sequence $S_k^s(m)$ has the following properties:

(i) $S_k^s(m)$ is composed of $N_k^s(m)$ symbols.

(ii) $S_k^s(m)$ is the concatenation of $F_k^s(m)$ fans of size m, where each fan is composed of m distinct symbols $a_1, a_2, \cdots, a_m$ and has the form

$$(a_1 a_2 \cdots a_{m-1} a_m a_{m-1} \cdots a_2 a_1),$$

so its length is $2m-1$.

(iii) $S_k^s(m)$ is a Davenport Schinzel sequence of order s.

S^*, and all the symbols in half of a fan are distinct. Thus, between two a's from the sequence S', there can only be one occurrence of b. We can thus get at worst the alternating subsequence

$$b \quad a \quad b \quad a \quad b.$$

We must now show property (iv): if every fan in $S_k^s(m)$ is replaced by some Davenport Schinzel sequence of order $s-2$ on the same m symbols with their first appearances remaining in the same order, then the result is still a Davenport Schinzel sequence of order s. We first show that no two adjacent elements are the same. For this, it suffices to show that the first element of every fan is not contained in the preceding fan. We show this by induction. It is clearly true for $S_1^s(m)$, $S_k^2(m)$, and $S_k^s(1)$. The first symbol in a fan in $S_k^s(m)$, $m > 1$, is the first symbol in the corresponding fan of the copy of $S_k^s(m-1)$ that it came from. The preceding fan either contains symbols from the previous copy of $S_k^s(m-1)$ or from the same copy of $S_k^s(m-1)$. In the first case, the two fans share no symbols from S'. In the second case, the first symbol of the fan is not in the preceding fan by our induction hypothesis (the preceding fan has been extended by an element of S^*, not of S'). Thus, when every fan is replaced by a sequence of order $s-2$, two adjacent elements from different sequences of order $s-2$ cannot be the same. Two adjacent elements within a sequence cannot be the same by the definition of a Davenport Schinzel sequence. Thus, no two adjacent elements are the same.

We must now show that when the fans are replaced, no alternating sequence

$$a \quad b \ldots a \quad b$$

of length $s+2$ appears. Suppose that such a sequence appears among the elements of S'. It must appear within one copy of $S_k^s(m-1)$, because different copies contain distinct symbols and do not interleave. If we delete the symbols from S^* in this copy of $S_k^s(m-1)$ and combine equal elements that have become adjacent, we obtain a copy of $S_k^s(m-1)$ where all the fans have been replaced by Davenport Schinzel sequences of order $s-2$, with symbols making their first appearance in the proper order. Such a sequence cannot contain a subsequence of length $s+2$ because, by condition (iv) applied to $S_k^s(m-1)$, it is a Davenport Schinzel sequence of order s.

When all the fans are replaced by Davenport Schinzel sequences of order $s-2$, a subsequence

$$a \quad b \ldots a \quad b$$

(iv) If every fan in $S_k^s(m)$ is replaced by any Davenport Schinzel sequence of order $s-2$ on the same m symbols, with the first appearances of these symbols in the same order, the resulting sequence is still a Davenport Schinzel sequence of order s.

Theorem 5.4 *$S_k^s(m)$ satisfies conditions (i)-(iv).*

Proof: By induction on s, k and m. Assume that they hold for $S_{k'}^{s'}(m')$ with $s' < s$, with $s' = s$ and $k' < k$, or with $s' = s$, $k' = k$ and $m' < m$. Checking properties (i) and (ii) is straightforward from the definition of the functions $N_k^s(m)$ and $F_k^s(m)$. Properties (iii) and (iv) obviously hold when $S_k^s(m) = (1\ 2\ \cdots\ m-1\ m\ m-1\ \cdots\ 2\ 1)$, which is the case when $S_k^s(m)$ is generated by method I or II in the definition. We must then prove that (iii) and (iv) hold when $S_k^s(m)$ is generated by method III or IV.

Consider first the case where $S_k^s(1)$ is obtained by method III. We must first show that $S_k^s(1)$ is a Davenport Schinzel sequence of order s. This is true because, by our induction hypothesis, property (iv) holds for the sequence $S_{k-1}^s(N_{k-1}^{s-2}(2^{k-1}))$. The sequence $S_k^s(1)$ is obtained by replacing every fan of this sequence by a sequence of order $s-2$. Property (iv) for $S_k^s(1)$ follows trivially form property (iii) because all fans have size 1.

Consider next the case of $S_k^s(m)$, when obtained by method IV. We first show that it is a Davenport Schinzel sequence of order s. It is easy to check that the same symbol cannot occur twice in a row, because the duplicated symbols in S' and $S^\star$ always have an element of the other sequence placed between them. It remains to check that $S_k^s(m)$ contains no subsequence

$$\underbrace{a\ \ldots\ b\ \ldots\ a\ \ldots\ b\ \ldots}_{s+2}$$

of length $s+2$, where $a \neq b$. If a and b are both from $S^\star$ there is no such alternating subsequence because, by property (iv) applied to the sequence $S_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1)))$, $S^\star$ is a Davenport Schinzel sequence of order s with some elements duplicated. If a and b are both from S', there is no subsequence of length $s+2$ because S' is the concatenation of Davenport-Schinzel sequences of order s on pairwise disjoint sets of symbols, again with some elements duplicated. This leaves the case when a belongs to S' and b to $S^\star$ (or vice versa; the proof for both cases is the same). We are safe here too because into each copy of the sequence $S_{k-1}^s(m-1)$ contained in S', we have only inserted either the *ascending* or the *descending* half of a fan of

of length $s+2$ cannot appear among the elements of $S^\star$ because each fan of $S_k^s(m)$ contains only one element of $S^\star$. Replacing each fan by a subsequence containing the same symbols thus cannot introduce any new alternations among the elements of $S^\star$.

We must still show that we cannot have a bad subsequence

$$a \qquad b \ldots a \qquad b$$

of length $s+2$ when a belongs to S' and b to $S^\star$ (the argument for the reverse situation is identical). Recall that for each copy of $S_k^s(m-1)$ in S', a different symbol of $S^\star$ is added to each fan. Since the only appearances of a are in a single copy of $S_k^s(m-1)$, we can restrict our attention to this copy and assume (in the worst case) that b's occur on both sides of it. The symbol b can appear in only one fan of this copy. After this fan is replaced by a Davenport Schinzel sequence of order $s-2$, this sequence will contain at worst a subsequence

$$\underbrace{a \ldots b \ldots a \ldots b \ldots a \ldots}_{s-1}$$

of length $s-1$ (a appears first because b was the middle element of the fan). There may be a's before and after this fan within the copy of $S_k^s(m-1)$, and b's before and after this copy of $S_k^s(m-1)$. We thus get, at worst, the alternating subsequence

$$b \qquad a \qquad (a \qquad b \qquad a \ldots b \qquad a) \qquad a \qquad b,$$

of length $s+1$. Property (iv) thus holds for $S_k^s(m)$.

□

Remark: The above proof fails for odd values of s. In particular, the last argument depends crucially on s being even, so that the alternating sequence of length $s-1$ starts and ends with a.

Theorem 5.5 *When* $n \geq A(7)$,

$$\lambda_s(n) \geq n \cdot 2^{K_s \alpha(n)^{\frac{s-2}{2}} + Q_s(n)}$$

where $K_s = \frac{1}{\frac{s-2}{2}!}$ *and* $Q_s(n)$ *is a polynomial in* $\alpha(n)$ *of degree at most* $\frac{s-4}{2}$.

Proof: Let $n_k^s = N_k^s(1)$. Then, for $k \geq 7$, we have

$$\begin{aligned} n_k^s &= N_k^s(1) \leq F_k^s(1) \\ &\leq F_k^\omega(1) \leq A_k(7) \\ &\leq A(k). \end{aligned}$$

We first show that $N_k^s(1) > N_{k-1}^s(1)$, since

$$N_k^s(1) = N_{k-1}^s(N_{k-1}^{s-2}(2^{k-1})) > N_{k-1}^s(1)$$

Thus, for any n, we can find k such that

$$n_k^s \leq n < n_{k+1}^s$$

Put $t = \left\lfloor \frac{n}{n_k^s} \right\rfloor$, so

$$t \cdot n_k^s \leq n < (t+1) \cdot n_k^s < 2t \cdot n_k^s$$

Now, using Lemma 5.1

$$\begin{aligned} \lambda_s(n) &\geq t \cdot \lambda_s(n_k^s) \geq t \cdot |S_k^s(1)| \\ &= t \cdot F_k^s(1) \\ &\geq t \cdot N_k^s(1) \cdot 2^{P(k,s)} \\ &> n \cdot 2^{P(k,s)-1} \end{aligned}$$

The definition of $P(k, s)$ gives

$$P(\alpha(n)-1, s) - 1 = K_s \cdot \alpha(n)^{\frac{s-2}{2}} + Q_s(n)$$

where Q_s is a polynomial in $\alpha(n)$ of degree at most $\frac{s-4}{2}$ and $K_s = \frac{1}{\frac{s-2}{2}!}$.

If $n \geq A(7)$, then we have $\alpha(n) \leq \alpha(n_{k+1}^s) \leq k+1$. Since P is an increasing function of k, this gives

$$\begin{aligned} \lambda_s(n) &\geq n \cdot 2^{P(\alpha(n)-1,s)-1} \\ &= n \cdot 2^{K_s \alpha(n)^{\frac{s-2}{2}} + Q_s(n)} \end{aligned}$$

□

References

[At] M. Atallah, *Dynamic Computational Geometry*, *Proc. 24th Symp. on Foundations of Computer Science*, 1983, pp. 92-99. (Also in *Comp. and Math. with Appls.* 11 (1985), pp. 1171-1181.)

[BS] A. Balstan and M. Sharir, *On shortest paths between two convex polyhedra*, Tech. Rept. 180, Dept. of Computer Science, New York University, 1985.

[Cl] K. Clarkson, *Approximate algorithms for shortest path motion planning*, *Proc.* 19^{th} *Annual ACM Symposium on Theory of Computing*, 1987, pp. 56-65.

[CS] R. Cole and M. Sharir, *Visibility problems for polyhedral terrains*, Tech. Rept. 266, Dept. of Computer Science, New York University, 1986.

[DS] H. Davenport and A. Schinzel, *A Combinatorial Problem Connected with Differential Equations*, *Amer. J. Math.*, 87 (1965), pp. 684-689.

[Da] H. Davenport, *A Combinatorial Problem Connected with Differential Equations II*, *Acta Arithmetica*, 17 (1971) pp. 363-372.

[ES] H. Edelsbrunner and M. Sharir, *The maximum number of ways to stab n convex non-intersecting objects in the plane is* $2n-2$, Tech. Rept. 281, New York University, 1986.

[GS] L. Guibas and M. Sharir, *The complexity of many faces in arrangement of lines and of segments*, In Preperation, 1987.

[GSS] L. Guibas, M. Sharir and S. Sifrony, *On the general motion planning problem with two degrees of freedom*, Manuscript, 1987.

[HS] S. Hart and M. Sharir, *Nonlinearity of Davenport-Schinzel Sequences and of Generalized Path Compression Schemes*, *Combinatorica*, 6 (1986) pp. 151-177.

[KS] K. Kedem and M. Sharir, *An efficient motion planning algorithm for a convex polygonal object in two-dimensional polygonal space*, Tech. Rept. 253, Dept. of Computer Science, New York University, 1986.

[LS] D. Leven and M. Sharir, *On the number of critical free contacts of a convex polygonal objects moving in 2-D polygonal space*, *Discrete and Computational Geometry*, 2 (1987), pp. 255-270.

[OSY] C. Ó'Dúnlaing , M. Sharir and C. Yap, *Generalized Voronoi diagrams for a ladder: II. Efficient construction of the diagram*, *Algorithmica*, 2 (1987) pp. 27-59.

[PS] J. Pach and M. Sharir, *The upper envelope of piecewise linear functions and the boundary of a region enclosed by convex planes*, Tech. Rept. 279, Dept. of Computer Science, New York University, 1987.

[PSS] R. Pollack, M. Sharir and S. Sifrony, *Separating two simple polygons by a sequence of translations*, Tech. Rept. 215, Dept. of Computer Science, New York University, 1986.

[RS] D. P. Roselle and R. G. Stanton, *Some Properties of Davenport-Schinzel Sequences*, *Acta Arithmetica*, 17 (1971) pp. 355-362.

[SS] J. Schwartz and M. Sharir, *On the two dimensional Davenport-Schinzel Problem*, Manuscript, 1987.

[Sh1] M. Sharir, *Almost Linear Upper Bounds on the Length of General Davenport-Schinzel Sequences*, *Combinatorica*, 7 (1987) pp. 131-143.

[Sh2] M. Sharir, *Improved Lower Bounds on the Length of Davenport-Schinzel Sequences*, Tech. Report 204, Dept. of Computer Science, New York University, April 1985 (to appear in *Combinatorica*).

[Sz] E. Szemerédi, *On a problem by Davenport and Schinzel*, *Acta Arithmetica*, 25 (1974) pp. 213-224.

[WS] A. Wiernik and M. Sharir, *Planar Realizations of nonlinear Davenport-Schinzel Sequences by Segments*, Tech. Rept. 224, Dept. of Computer Science, New York University, June 1986 (to appear in *Discrete and Computational Geometry*).

Appendix 1:

In this appendix, we give the proofs of Lemmas 2.4 - 2.8.

Lemma 2.4 *For all $k \geq 1$, $A_k(2) = 4$ and $A_k(3) \geq 2k$.*

Proof: First consider $A_k(2)$. For $k = 1$, $A_1(2) = 2 \times 2 = 4$. By induction,

$$\begin{aligned} A_{k+1}(2) &= A_k(A_{k+1}(1)) \\ &= A_k(2) = 4. \end{aligned}$$

As to $A_k(3)$, For $k = 1$, $A_1(3) = 6 > 2$.
For $k = 2$, $A_2(3) = 8 > 2 \times 2$.
For $k > 2$, assume the inequality is true for all $k' < k$, then

$$\begin{aligned} A_k(3) &= A_{k-1}(A_k(2)) \\ &= A_{k-1}(4) = A_{k-2}(A_{k-1}(3)) \\ &> A_{k-2}(2(k-1)) \geq 4(k-1) \\ &> 2k \end{aligned}$$

□

Lemma 2.5 *For all $n \geq 1$, $\alpha_{\alpha(n)+1}(n) \leq 4$.*

Proof: By the definition of $\alpha_k(n)$,

$$\begin{aligned} \alpha_{\alpha(n)+1}(n) &= \min\{\, s \geq 1 \,:\, \alpha^{(s)}_{\alpha(n)}(n) = 1\} \\ &= \min\{\, s \geq 1 \,:\, \alpha^{(s)}_{\alpha(n)}(\alpha(n)) = 1\} + 1 \end{aligned}$$

By Lemma 2.4, $\alpha_{\alpha(n)}(\alpha(n)) \leq 3$, therefore after applying $\alpha_{\alpha(n)}$ once more:

$$\alpha_{\alpha(n)+1}(n) \leq \min\{\, s \geq 1 \,:\, \alpha^{(s)}_{\alpha(n)}(4) = 1\} + 2$$

But, by Lemma 2.4, $\alpha_k(4) = 2$. Therefore

$$\alpha_{\alpha(n)+1}(n) \leq 4$$

□

Lemma 2.6 *For all $k \geq 4$ and $s \geq 3$,*

$$2^{A_k(s)} \leq A_{k-1}(\log(A_k(s)))$$

Proof:

$$\begin{aligned} A_{k-1}(\log(A_k(s))) &= A_{k-2}(A_{k-1}(\log(A_k(s)) - 1)) \\ &= A_{k-2}(A_{k-2}(A_{k-1}(\log(A_k(s)) - 2))) \\ &\geq A_2(A_2(2^{\log(A_k(s))-2})) \\ &= A_2(2^{\frac{A_k(s)}{4}}) \end{aligned}$$

For $x \geq 16$, $2^{\frac{x}{4}} \geq x$. For $k \geq 3$ and $s \geq 3$, $A_k(s) \geq 2^{2^2} = 16$. Therefore

$$A_{k-1}(\log(A_k(s))) \geq A_2(A_k(s)) = 2^{A_k(s)}$$

□

Lemma 2.7 *Let $\xi_k(n)$ be $2^{\alpha_k(n)}$. Then for $k \geq 3$, $n \geq A_{k+1}(4)$*

$$\min\{s' \geq 1 : \xi_k^{(s')}(n) \leq A_{k+1}(4)\} \leq 2 \cdot \alpha_{k+1}(n) - 2$$

Proof: We first prove it for n having the form $n = A_{k+1}(q)$ by induction on q. It is obvious for $n = A_{k+1}(4)$ as the left hand side is 1. Let us assume it is true for all $q' \leq q$. Now consider $n = A_{k+1}(q+1)$.

$$\begin{aligned} & \min\{s' \geq 1 : \xi_k^{(s')}(A_{k+1}(q+1)) \leq A_{k+1}(4)\} \\ = \; & \min\{s' \geq 1 : \xi_k^{(s')}(A_k(A_{k+1}(q))) \leq A_{k+1}(4)\} \\ = \; & \min\{s' \geq 1 : \xi_k^{(s')}(2^{A_{k+1}(q)}) \leq A_{k+1}(4)\} + 1 \\ \leq \; & \min\{s' \geq 1 : \xi_k^{(s')}(A_k(\log(A_{k+1}(q)))) \leq A_{k+1}(4)\} + 1 \end{aligned}$$

$$
\begin{aligned}
& \text{(Using Lemma 2.6)} \\
= \quad & \min\{\, s' \geq 1 \;:\; \xi_k^{(s')}(A_{k+1}(q)) \;\leq\; A_{k+1}(4)\} \;+2 \\
\leq \quad & 2 \cdot \alpha_{k+1}(A_{k+1}(q)) \;-2 \;+2 \\
& \text{(By inductive hypothesis)} \\
= \quad & 2q - 2 + 2 = 2 \cdot (q+1) - 2 \\
= \quad & 2 \cdot \alpha_{k+1}(A_{k+1}(q+1)) - 2
\end{aligned}
$$

For general values of n,

$$A_{k+1}(\alpha_{k+1}(n) - 1) \;<\; n \leq A_{k+1}(\alpha_{k+1}(n))$$

and also $\alpha_{k+1}(n) = \alpha_{k+1}(A_{k+1}(\alpha_{k+1}(n)))$. Therefore

$$
\begin{aligned}
& \min\{\, s' \geq 1 \;:\; \xi_k^{(s')}(n) \;\leq\; A_{k+1}(4)\} \\
\leq \quad & \min\{\, s' \geq 1 \;:\; \xi_k^{(s')}(A_{k+1}(\alpha_{k+1}(n))) \;\leq A_{k+1}(4)\} \\
= \quad & 2 \cdot \alpha_{k+1}(A_{k+1}(\alpha_{k+1}(n))) - 2 \\
= \quad & 2 \cdot \alpha_{k+1}(n) - 2
\end{aligned}
$$

□

Lemma 2.8 *For all $k \geq 1$, $n \geq 2$, $\beta_k(n) \leq 2\alpha_k(n)$.*

Proof: For $k \leq 2$, it follows directly from the definition of $\beta_k(n)$. For $k = 3$

$$\beta_3(n) \;=\; \min\{\, s' \geq 1 \;:\; (\alpha_2 \cdot \alpha_2)^{(s')}(n) \;\leq\; 64\}$$

We first prove this for n of the form $A_3(q)$. For $n = A_3(2) = 4$, and $n = A_3(3) = 16$ it is true as $\beta_3(n)$ is simply 1. Assume that it is true for some $q \geq 3$; then

$$
\begin{aligned}
\beta_3(A_3(q+1)) \quad &= \quad \min\{\, s' \geq 1 \;:\; (\log^2)^{(s')}(A_3(q+1)) \;\leq\; 64\} \\
&= \quad \min\{\, s' \geq 1 \;:\; (\log^2)^{(s')}(A_2(A_3(q))) \;\leq\; 64\} \\
&= \quad \min\{\, s' \geq 1 \;:\; (\log^2)^{(s')}(A_3(q) \cdot A_3(q)) \;\leq\; 64\} \;+1 \\
&= \quad \min\{\, s' \geq 1 \;:\; (\log^2)^{(s')}(4 \log^2 A_3(q)) \;\leq\; 64\} \;+2 \cdot
\end{aligned}
$$

For $q = 3$, $A_3(q) = 16$ and therefore $4 \log^2 A_3(q) = 64$, which implies

$$\beta_3(A_3(q+1)) \;=\; 3 \;\leq\; 2 \cdot \alpha_3(A_3(q)).$$

For $q > 3$, $\log A_3(q) \geq 16$ and for $x \geq 16$, $4x^2 \leq 2^x$; therefore

$$\begin{aligned}
\beta_3(A_3(q+1)) &\leq \min\{\, s' \geq 1 \,:\, (\log^2)^{(s')}(A_3(q)) \,\leq\, 64\} \,+2 \\
&= \beta_3(A_3(q)) + 2 \\
&\leq 2\alpha_3(A_3(q)) + 2 \\
&= 2\alpha_3(A_3(q+1))
\end{aligned}$$

For general values of n,

$$A_3(\alpha_3(n)-1) \,<\, n \,\leq\, A_3(\alpha_3(n))$$

and $\alpha_3(n) = \alpha_3(A_3(\alpha_3(n)))$. Using the same argument as in the previous lemma we can show that $\beta_3(n) \leq 2\alpha_3(n)$.

For $k > 3$, $n \leq A_k(4) = A_{k-1}(A_k(3))$, we have $\alpha_{k-1}(n) \leq A_k(3)$ and by induction hypothesis $\beta_{k-1}(n) \leq 2A_k(3)$. Hence

$$\alpha_{k-1}(n) \cdot \beta_{k-1}(n) \,\leq\, 2A_k^2(3).$$

But for $k > 3$, $A_k(3) \geq 8$ and for $x \geq 8$, $2x^2 \leq 2^x$, hence

$$\begin{aligned}
q &\equiv \alpha_{k-1}(n) \cdot \beta_{k-1}(n) \\
&\leq 2^{A_k(3)} \;=\; 2^{A_{k-1}(A_k(2))} \\
&= 2^{A_{k-1}(4)} \;\; \text{(Using Lemma 2.4)} \\
&\leq A_{k-2}(A_{k-1}(4)) \;=\; A_{k-1}(5)
\end{aligned}$$

and therefore

$$\alpha_{k-1}(q) \cdot \beta_{k-1}(q) \,\leq\, 5 \times 10 \,<\, 64$$

Thus for $n \leq A_k(4)$, $\beta_k(n) \leq 2$ which clearly implies the assertion. For $n > A_k(4) = A_{k-1}(A_k(3))$, we have

$$\begin{aligned}
\beta_k(n) &= \min\{\, s' \geq 1 \,:\, (\alpha_{k-1} \cdot \beta_{k-1})^{(s')}(n) \,\leq\, 64\} \\
&= \min\{\, s' \geq 1 \,:\, q \equiv (\alpha_{k-1} \cdot \beta_{k-1})^{(s')}(n) \,\leq\, A_k(4)\} \,+ \\
&\quad\; \min\{t \geq 1 \,:\, (\alpha_{k-1} \cdot \beta_{k-1})^{(t)}(q) \,\leq\, 64\}
\end{aligned}$$

By induction hypothesis,

$$\alpha_{k-1}(n') \cdot \beta_{k-1}(n') \,\leq\, \alpha_{k-1}(n') \cdot 2\alpha_{k-1}(n')$$

But as long as $n' > A_k(4)$, we have $\alpha_{k-1}(n') \geq A_k(3) \geq 8$, so that

$$\begin{aligned} \alpha_{k-1}(n') \cdot \beta_{k-1}(n') &\leq 2\alpha_{k-1}^2(n') \\ &\leq 2^{\alpha_{k-1}(n')} = \xi_{k-1}(n') \end{aligned}$$

Therefore

$$\begin{aligned} \beta_k(n) &\leq \min\{ s' \geq 1 : q' \equiv \xi_{k-1}^{(s')}(n) \leq A_k(4)\} + \\ &\quad \min\{t \geq 1 : (\alpha_{k-1} \cdot \beta_{k-1})^{(t)}(q') \leq 64\} \\ &\leq 2 \cdot \alpha_k(n) - 2 + \beta_k(A_k(4)) \\ &\leq 2 \cdot \alpha_k(n) - 2 + 2 \\ &= 2\alpha_k(n) \end{aligned}$$

$\square$

Appendix 2:

In this appendix, we provide the proofs of properties (P.3)-(P.5) of the functions $F_k(m)$.

(P.3) $\{F_k(m)\}_{k\geq 1}^{\infty}$ is strictly increasing for a fixed $m \geq 1$.

Proof: Indeed, for $k = 2$, $F_2(m) = 3 \cdot 2^{m-1} \geq 1 = F_1(m)$. For $k > 2$ and $m = 1$

$$\begin{aligned} F_k(1) &= (2^k - 1) \cdot F_{k-1}(2^{k-1}) \\ &> F_{k-1}(2^{k-1}) \\ &> F_{k-1}(1) \end{aligned}$$

Now for $k > 2$, $m \geq 1$, assume the assertion is true for all $k' < k$ and for $k' = k$ and $m' < m$. Then we have

$$\begin{aligned} F_k(m) &= 2F_k(m-1) \cdot F_{k-1}(F_k(m-1)) \\ &\geq 2F_k(m-1) \cdot F_{k-1}(m) \\ &> F_{k-1}(m) \end{aligned}$$

□

(P.4) $F_k(m) \geq A_k(m)$ for $k \geq 2$, $m \geq 1$.

Proof: Indeed, for $k = 2$, the assertion is true by (P.1), for $k > 2$, $m = 1$,

$$\begin{aligned} F_k(1) &= (2^k - 1) \cdot F_{k-1}(2^{k-1}) \geq 2 \\ &= A_k(1) \end{aligned}$$

For $k \geq 2$, $m \geq 1$, assume it is true for all $k' < k$, and all $k' = k$ and $m' < m$. We have

$$\begin{aligned} F_k(m) &= 2F_k(m-1) \cdot F_{k-1}(F_k(m-1)) \\ &\geq F_{k-1}(F_k(m-1)) \\ &\geq A_{k-1}(A_k(m-1)) \\ &\geq A_k(m) \end{aligned}$$

□

(P.5) $\quad 2^{F_k(m)} \leq A_k(m+4) \qquad$ for $k \geq 3$, $m \geq 1$.

Proof: Indeed,

$$\begin{aligned} F_3(1) &= (2^3-1)\cdot F_2(2^2) \\ &\leq 2^3 \times 3 \times 2^4 \\ &\leq 2^{3+2+4} \leq 2^{16} \\ &= A_3(4) \end{aligned}$$

Therefore $2^{F_3(1)} \leq 2^{A_3(4)} = A_3(5)$.
For $k = 3$ and $m > 1$, assume the assertion is true for all $m' < m$, then

$$\begin{aligned} F_3(m) &= 2F_3(m-1)\cdot F_2(F_3(m-1)) \\ &\leq F_2(2F_3(m-1)\cdot F_3(m-1)) \\ &\leq 2^2 \cdot 2^{2F_3(m-1)\cdot F_3(m-1)} \\ &\leq 2^{2F_3(m-1)\cdot F_3(m-1)+2} \\ &\leq 2^{4F_3(m-1)\cdot F_3(m-1)} \end{aligned}$$

$F_3(m-1) \geq 8$, therefore $4F_3(m-1)\cdot F_3(m-1) \leq 2^{F_3(m-1)}$. Thus

$$\begin{aligned} F_3(m) &\leq 2^{2^{F_3(m-1)}} \leq 2^{A_3(m+3)} \\ &= A_3(m+4) \end{aligned}$$

Now for $k > 3$ and $m = 1$,

$$\begin{aligned} F_k(1) &= (2^k-1)\cdot F_{k-1}(2^{k-1}) \leq 2^k \cdot F_{k-1}(2^{k-1}) \\ &\leq F_{k-1}(2^k \cdot 2^k) \leq F_{k-1}(2^{2k}) \end{aligned}$$

Therefore,

$$2^{F_k(1)} \leq 2^{F_{k-1}(2^{2k})} \leq A_{k-1}(2^{2k}+4)$$

But for $k \geq 4$,

$$\begin{aligned} A_k(4) &\geq A_3(A_k(3)) \geq A_3(2k) \qquad \text{(By Lemma 2.4)} \\ &= A_2(A_3(2k-1)) \geq A_2(2(2k-1)) \end{aligned}$$

For $k \geq 4$, $2(2k-1) > 2k+1$, so

$$\begin{aligned} A_k(4) &\geq A_2(2k+1) = 2^{2k+1} \\ &\geq 2^{2k} + 4 \end{aligned}$$

and thus

$$2^{F_k(1)} \leq A_{k-1}(A_k(4)) \leq A_k(5)$$

Finally for $k > 3$ and $m > 1$, assume the assertion is true for all $k' < k$ and $m \geq 1$ and for $k' = k$ and $m' < m$, then

$$\begin{aligned} F_k(m) &= 2F_k(m-1) \cdot F_{k-1}(F_k(m-1)) \\ &\leq F_{k-1}(2F_k(m-1) \cdot F_k(m-1)) \end{aligned}$$

Thus
$$\begin{aligned} 2^{F_k(m)} &\leq 2^{F_{k-1}(2F_k(m-1)\cdot F_k(m-1))} \\ &\leq A_{k-1}(2F_k(m-1) \cdot F_k(m-1) + 4) \\ &\leq A_{k-1}(2^{F_k(m-1)}) \qquad \text{(because } F_k(m-1) \geq F_4(1) \geq 15) \\ &\leq A_{k-1}(A_k(m+3)) \\ &\leq A_k(m+4) \end{aligned}$$

$\square$

Appendix 3:

In this appendix, we provide proofs of some properties of the auxiliary functions used in obtaining the lower bounds for $\lambda_s(n)$.

Fact 5.3: For $k \geq 2$, $D_k \leq 2^{-(k-2)}$.

Proof: We prove this by

$$\begin{aligned}
\prod_{j=1}^{k} \frac{2^{j-1}}{2^j - 1} &\leq 1 \cdot \prod_{j=2}^{k} \frac{2^j - 1}{2^{j+1} - 4} \\
&= \frac{\prod_{j=2}^{k} (2^j - 1)}{\prod_{j=2}^{k} (2^{j+1} - 4)} \\
&= \frac{\prod_{j=2}^{k} (2^j - 1)}{4^{k-1} \cdot \prod_{j=1}^{k-1} (2^j - 1)} \\
&= \frac{2^k - 1}{4^{k-1}} \\
&\leq 2^{-k+2}.
\end{aligned}$$

□

Fact 5.4: For $k \geq 2$,

$$P(k, s) = \sum_{i=1}^{k-1} P(i, s-2) + k - 2.$$

Proof: We prove this by induction. It is clearly true when $k = 2$, since all terms in the summation are zero. Now, assume this is true for all $k' < k$. Then

$$\begin{aligned}
\sum_{i=1}^{k-1} P(i, s-2) + k - 2 &= 1 + P(k-1, s-2) + \sum_{i=1}^{k-2} P(i, s-2) + k - 3 \\
&= 1 + P(k-1, s-2) + P(k-1, s) \\
&= 1 + \sum_{i=1}^{\frac{s}{2}-2} \binom{k-3}{i} + \sum_{i=1}^{\frac{s}{2}-1} \binom{k-3}{i}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\frac{s}{2}-1}\left(\binom{k-3}{i-1}+\binom{k-3}{i}\right) \\
&= \sum_{i=1}^{\frac{s}{2}-1}\binom{k-2}{i} \\
&= P(k,s)
\end{aligned}
$$

which is what we were trying to prove.

□

Fact 5.8: For $k \geq 2$, $A_k(m+1) \geq 2A_k(m)$.

Proof: This is clear for $k = 2$. For $k \geq 3$, we assume it is true for smaller k. This gives

$$
\begin{aligned}
A_k(m+1) &= A_{k-1}(A_k(m)) \\
&\geq 2A_{k-1}(A_k(m)-1) \\
&\geq 2A_{k-1}(A_k(m-1)) \\
&= 2A_k(m)
\end{aligned}
$$

.

□

Lemma 5.2 *For all s, $F_k^\omega(m) \geq F_k^s(m)$.*

Proof: We proceed using induction, Facts 5.1 and 5.2, and the inequality $m \cdot F_k^s(m) \geq N_k^s(m)$, which follows from Lemma 5.1.
For $k = 1$ or $s = 2$, the theorem is trivial since $F_1^s(m) = F_k^2(m) = 1$.

We now assume that we have shown this for smaller s, for smaller k with the same s, and for smaller m with the same k and s. For $m = 1$, we have

$$
\begin{aligned}
F_k^s(1) &= (2^k-1)\cdot F_{k-1}^{s-2}(2^{k-1})\cdot F_{k-1}^s(N_{k-1}^{s-2}(2^{k-1})) \\
&\leq (2^k-1)\cdot F_{k-1}^\omega(2^{k-1})\cdot F_{k-1}^s(2^{k-1}\cdot F_{k-1}^{s-2}(2^{k-1})) \\
&\leq (2^k-1)\cdot F_{k-1}^\omega(2^{k-1})\cdot F_{k-1}^s(2^{k-1}\cdot F_{k-1}^\omega(2^{k-1})) \\
&\leq (2^k-1)\cdot F_{k-1}^\omega(2^{k-1})\cdot F_{k-1}^\omega(2^{k-1}\cdot F_{k-1}^\omega(2^{k-1})) \\
&= F_k^\omega(1)
\end{aligned}
$$

Similarly, for $k, m \geq 1$, we have

$$\begin{aligned}
F_k^s(m) &= 2F_k^s(m-1) \cdot F_{k-1}^{s-2}(F_k^s(m-1)) \cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^s(m-1))) \\
&\leq 2F_k^\omega(m-1) \cdot F_{k-1}^{s-2}(F_k^\omega(m-1)) \cdot F_{k-1}^s(N_{k-1}^{s-2}(F_k^\omega(m-1))) \\
&\leq 2F_k^\omega(m-1) \cdot F_{k-1}^\omega(F_k^\omega(m-1)) \cdot F_{k-1}^s(F_k^\omega(m-1) \cdot F_{k-1}^{s-2}(F_k^\omega(m-1))) \\
&\leq 2F_k^\omega(m-1) \cdot F_{k-1}^\omega(F_k^\omega(m-1)) \cdot F_{k-1}^\omega(F_k^\omega(m-1) \cdot F_{k-1}^\omega(F_k^\omega(m-1))) \\
&= F_k^\omega(m)
\end{aligned}$$

□

Lemma 5.3

$$F_k^\omega(m) \leq A_k(7m)$$

Proof: We first show several horrible inequalities on F^ω. Using Facts 5.5 through 5.7 extensively, we see that for $k \geq 3$,

$$\begin{aligned}
F_k^\omega(1) &= (2^k - 1) \cdot F_{k-1}^\omega(2^{k-1}) \cdot F_{k-1}^\omega(2^{k-1} \cdot F_{k-1}^\omega(2^{k-1})) \\
&\leq F_{k-1}^\omega(k + 2^{k-1}) \cdot F_{k-1}^\omega(F_{k-1}^\omega(k - 1 + 2^{k-1})) \\
&\leq F_{k-1}^\omega(2 \cdot F_{k-1}^\omega(k + 2^{k-1})) \\
&\leq F_{k-1}^\omega(F_{k-1}^\omega(k + 1 + 2^{k-1})) \\
&\leq F_{k-1}^\omega(F_{k-1}^\omega(2^k)) \\
&\leq F_{k-1}^\omega(F_{k-1}^\omega(F_{k-1}^\omega(k))) \\
&\leq F_{k-1}^\omega(F_{k-1}^\omega(F_{k-1}^\omega(F_{k-1}^\omega(1))))
\end{aligned}$$

The last step follows since

$$F_{k-1}^\omega(1) \geq 2^{k-1} - 1 \geq k \text{ for } k \geq 3.$$

Similarly, for $k \geq 3$,

$$\begin{aligned}
F_k^\omega(m) &= 2 \cdot F_k^\omega(m-1) \cdot F_{k-1}^\omega(F_k^\omega(m-1)) \cdot F_{k-1}^\omega(F_k^\omega(m-1) \cdot F_{k-1}^\omega(F_k^\omega(m-1))) \\
&\leq 2 \cdot F_{k-1}^\omega(2 \cdot F_k^\omega(m-1)) \cdot F_{k-1}^\omega(F_{k-1}^\omega(2 \cdot F_k^\omega(m-1)))
\end{aligned}$$

$$
\begin{aligned}
&\le 2 \cdot F^{\omega}_{k-1}(2 \cdot F^{\omega}_{k-1}(2 \cdot F^{\omega}_{k}(m-1))) \\
&\le F^{\omega}_{k-1}(1 + 2 \cdot F^{\omega}_{k-1}(2 \cdot F^{\omega}_{k}(m-1))) \\
&\le F^{\omega}_{k-1}(4 \cdot F^{\omega}_{k-1}(2 \cdot F^{\omega}_{k}(m-1))) \\
&\le F^{\omega}_{k-1}(F^{\omega}_{k-1}(2 + 2 \cdot F^{\omega}_{k}(m-1))) \\
&\le F^{\omega}_{k-1}(F^{\omega}_{k-1}(4 \cdot F^{\omega}_{k}(m-1))) \\
&\le F^{\omega}_{k-1}(F^{\omega}_{k-1}(F^{\omega}_{k-1}(F^{\omega}_{k}(m-1))))
\end{aligned}
$$

This last step follows from the inequalities $F^{\omega}_{k-1}(x) \ge 2^x \ge 4x$ when $x \ge 4$, $k \ge 3$ and $F^{\omega}_{k}(x) \ge 2^k - 1 \ge 4$ when $k \ge 3$.

We will now show $F^{\omega}_{k}(m) \le A_k(7m)$. This is easy for $k = 1$ and $k = 2$. For $k \ge 3$, we assume that it is true for k', m' when $k' < k$ and when $k' = k$ and $m' < m$. This gives

$$
\begin{aligned}
F^{\omega}_{k}(1) &\le F^{\omega}_{k-1}(F^{\omega}_{k-1}(F^{\omega}_{k-1}(F^{\omega}_{k-1}(1)))) \\
&\le A_{k-1}(7 \cdot A_{k-1}(7 \cdot A_{k-1}(7 \cdot A_{k-1}(7)))) \\
&\le A_{k-1}(8 \cdot A_{k-1}(7 \cdot A_{k-1}(7 \cdot A_{k-1}(7)))) \\
&\le A_{k-1}(A_{k-1}(3 + 7 \cdot A_{k-1}(7 \cdot A_{k-1}(7)))) \\
&\le A_{k-1}(A_{k-1}(16 \cdot A_{k-1}(7 \cdot A_{k-1}(7)))) \\
&\le A_{k-1}(A_{k-1}(A_{k-1}(4 + 7 \cdot A_{k-1}(7)))) \\
&\le A_{k-1}(A_{k-1}(A_{k-1}(A_{k-1}(11)))) \\
&\le A_{k-1}(A_{k-1}(A_{k-1}(A_{k-1}(A_{k-1}(A_{k-1}(2)))))) \\
&= A_k(7)
\end{aligned}
$$

Similarly, for $m > 1$, we get

$$
\begin{aligned}
F^{\omega}_{k}(m) &\le F^{\omega}_{k-1}(F^{\omega}_{k-1}(F^{\omega}_{k-1}(F^{\omega}_{k}(m-1)))) \\
&\le A_{k-1}(7 \cdot A_{k-1}(7 \cdot A_{k-1}(7 \cdot A_k(7m-7)))) \\
&\le A_{k-1}(A_{k-1}(3 + 7 \cdot A_{k-1}(7 \cdot A_k(7m-7)))) \\
&\le A_{k-1}(A_{k-1}(A_{k-1}(4 + 7 \cdot A_k(7m-7)))) \\
&\le A_{k-1}(A_{k-1}(A_{k-1}(A_k(7m-3)))) \\
&= A_k(7m)
\end{aligned}
$$

□

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